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Robust Fuzzy Control of Parametric Uncertain Nonlinear Systems Using Robust Reliability Method

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1. Introduction

Stability is of primary importance for any control systems. Stability of both linear and nonlinear uncertain systems has received a considerable attention in the past decades (see for example, Tanaka & Sugeno, 1992; Tanaka, Ikeda, & Wang, 1996; Feng, Cao, Kees, et al. 1997; Teixeira & Zak, 1999; Lee, Park, & Chen, 2001; Park, Kim, & Park, 2001; Chen, Liu, & Tong, 2006; Lam & Leung, 2007, and references therein). Fuzzy logical control (FLC) has proved to be a successful control approach for a great many complex nonlinear systems. Especially, the well-known Takagi-Sugeno (T-S) fuzzy model has become a convenient tool for dealing with complex nonlinear systems. T-S fuzzy model provides an effective representation of nonlinear systems with the aid of fuzzy sets, fuzzy rules and a set of local linear models. Once the fuzzy model is obtained, control design can be carried out via the so called parallel distributed compensation (PDC) approach, which employs multiple linear controllers corresponding to the locally linear plant models (Hong & Langari, 2000). It has been shown that the problems of controller synthesis of nonlinear systems described by the T-S fuzzy model can be reduced to convex problems involving linear matrix inequalities (LMIs) (Park, Kim, & Park, 2001). Many significant results on the stability and robust control of uncertain nonlinear systems using T-S fuzzy model have been reported (see for example, Hong, & Langari, 2000; Park, Kim, & Park, 2001; Xiu & Ren, 2005; Wu & Cai, 2006; Yoneyama, 2006; 2007), and considerable advances have been made. However, as stated in Guo (2010), many approaches for stability and robust control of uncertain systems are often characterized by conservatism when dealing with uncertainties. In practice, uncertainty exists in almost all engineering systems and is frequently a source of instability and deterioration of performance. So, uncertainty is one of the most important factors that have to be taken into account rationally in system analysis and synthesis. Moreover, it has been shown (Guo, 2010) that the increasing in conservatism in dealing with uncertainties by some traditional methods does not mean the increasing in reliability. So, it is significant to deal with uncertainties by means of reliability approach and to achieve a balance between reliability and performance/control-cost in design of uncertain systems.

In fact, traditional probabilistic reliability methods have ever been utilized as measures of stability, robustness, and active control effectiveness of uncertain structural systems by Spencer et al. (1992,1994); Breitung et al. (1998) and Venini & Mariani (1999) to develop

robust control strategies which maximize the overall reliability of controlled structures. Robust control design of systems with parametric uncertainties have also been studied by Mengali and Pieracci (2000); Crespo and Kenny (2005). These works are meaningful in improving the reliability of uncertain controlled systems, and it has been shown that the use of reliability analysis may be rather helpful in evaluating the inherent uncertainties in system design. However, these works are within the probabilistic framework.

In Guo (2007,2010), a non-probabilistic robust reliability method for dealing with bounded parametric uncertainties of linear controlled systems has been presented. The non-probabilistic procedure can be implemented more conveniently than probabilistic one whether in dealing with the uncertainty data or in controller design of uncertain systems, since complex computations are often associated with in controller design of uncertain systems. In this chapter, following the basic idea developed in Guo (2007, 2010), we focus on developing a robust reliability method for robust fuzzy controller design of uncertain nonlinear systems.

2. Problem statements and preliminary knowledge

Consider a nonlinear uncertain system represented by the following T-S fuzzy model with parametric uncertainties:

Plant Rule i:

$$\begin{aligned} &\text{IF } x_1(t) \text{ is } F_{i1} \text{ and } \ldots \text{ and } x_n(t) \text{ is } F_{in}, \\ &\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\boldsymbol{\rho})\mathbf{x}(t) + \mathbf{B}_i(\boldsymbol{\rho})\mathbf{u}(t), \ (i = 1,\ldots,r) \end{aligned} \tag{1}$$

Where F_{ij} is a fuzzy set, $\boldsymbol{x}(t) \in R^n$ is the state vector, $\boldsymbol{u}(t) \in R^m$ is the control input vector. r is the number of rules of the T-S fuzzy model. The system matrices $\boldsymbol{A}(\boldsymbol{\rho})$ and $\boldsymbol{B}(\boldsymbol{\rho})$ depend on the uncertain parameters $\boldsymbol{\rho} = \{\rho_1, \rho_2, \cdots, \rho_p\}$.

The defuzzified output of the fuzzy system can be represented by

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \mu_i(\boldsymbol{x}(t))[\boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{\rho})\boldsymbol{u}(t)] \tag{2}$$

In which

$$\mu_i(\boldsymbol{x}(t)) = \omega_i(\boldsymbol{x}(t)) \Big/ \sum_{i=1}^{r} \omega_i(\boldsymbol{x}(t)), \ \omega_i(\boldsymbol{x}(t)) = \prod_{j=1}^{n} F_{ij}(x_j(t)) \tag{3}$$

Where $F_{ij}(x_j(t))$ is the grade of membership of $x_j(t)$ in the fuzzy set F_{ij}, $\omega_i(\boldsymbol{x}(t))$ satisfies $\omega_i(\boldsymbol{x}(t)) \geq 0$ for all i ($i = 1,\ldots,r$). Therefore, there exist the following relations

$$\mu_i(\boldsymbol{x}(t)) \geq 0 \ \ (i = 1,\ldots,r), \ \sum_{i=1}^{r} \mu_i(\boldsymbol{x}(t)) = 1 \tag{4}$$

If the system (1) is local controllable, a fuzzy model of state feedback controller can be stated as follows:

Control Rule i:

$$\text{IF } x_1(t) \text{ is } F_{i1} \text{ and } \ldots \text{ and } x_n(t) \text{ is } F_{in}, \text{ THEN } \boldsymbol{u}(t) = \boldsymbol{K}_i\boldsymbol{x}(t), \ (i = 1,\ldots,r) \tag{5}$$

Where $\boldsymbol{K}_i \in R^{m\times n}$ $(i=1,\ldots,r)$ are gain matrices to be determined. The final output of the fuzzy controller can be obtained by

$$\boldsymbol{u}(t)=\sum_{i=1}^{r}\mu_i(\boldsymbol{x}(t))\boldsymbol{K}_i\boldsymbol{x}(t) \tag{6}$$

By substituting the control law (6) into (2), we obtain the closed-loop system as follows

$$\dot{\boldsymbol{x}}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(\boldsymbol{x}(t))\mu_j(\boldsymbol{x}(t))[\boldsymbol{A}_i(\boldsymbol{\rho})+\boldsymbol{B}_i(\boldsymbol{\rho})\boldsymbol{K}_j]\boldsymbol{x}(t) \tag{7}$$

When the parameterized notation (Tuan, Apkarian, and Narikiyo 2001) is used, equations (6) and (7) can be rewritten respectively as

$$\boldsymbol{u}(t)=\boldsymbol{K}(\mu)\boldsymbol{x}(t) \tag{8}$$

$$\dot{\boldsymbol{x}}(t)=\big(\boldsymbol{A}(\boldsymbol{\rho},\mu)+\boldsymbol{B}(\boldsymbol{\rho},\mu)\boldsymbol{K}(\mu)\big)\boldsymbol{x}(t) \tag{9}$$

Where

$$\mu=\big(\mu_1(\boldsymbol{x}(t)),\ldots,\mu_r(\boldsymbol{x}(t))\big)\in\Omega:=\left\{\mu\in R^r:\sum_{i=1}^{r}\mu_i(\boldsymbol{x}(t))=1,\mu_i(\boldsymbol{x}(t))\geq 0\right\} \tag{10}$$

$$\boldsymbol{K}(\mu)=\sum_{i=1}^{r}\mu_i(\boldsymbol{x}(t))\boldsymbol{K}_i\ ,\ \boldsymbol{A}(\boldsymbol{\rho},\mu)=\sum_{i=1}^{r}\mu_i(\boldsymbol{x}(t))\boldsymbol{A}_i(\boldsymbol{\rho})\ ,\ \boldsymbol{B}(\boldsymbol{\rho},\mu)=\sum_{i=1}^{r}\mu_i(\boldsymbol{x}(t))\boldsymbol{B}_i(\boldsymbol{\rho}) \tag{11}$$

Note that the uncertain parameters $\boldsymbol{\rho}=\{\rho_1,\rho_2,\cdots,\rho_p\}$ are involved in the expressions of (9) and (11). Following the basic idea developed by Guo (2007,2010), the uncertain-but-bounded parameters $\boldsymbol{\rho}=\{\rho_1,\rho_2,\cdots,\rho_p\}$ involved in the problem can be considered as interval variables and expressed in the following normalized form

$$\rho_i=\rho_{i0}+\rho_{id}\delta_i\ \ (i=1,\ldots,p) \tag{12}$$

where ρ_{i0} and ρ_{id} are respectively the nominal and deviational values of the uncertain parameter ρ_i, $\delta_i\in[-1,1]$ is a standard interval variable. Furthermore, the system matrices are expressed in a corresponding form of that depend on the standard interval variables $\boldsymbol{\delta}=[\delta_1,\delta_2,\cdots,\delta_p]$. Suppose that the stability of the control system can be reduced to solving a matrix inequality as follows

$$\boldsymbol{M}(\boldsymbol{\delta},\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_l)<0 \tag{13}$$

where, $\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_l$ are feasible matrices to be determined. The sign "<0" indicates that the matrix is negative-definite.

If the performance function (it may also be referred to as limit-state function) used for reliability analysis is defined in terms of the criterion (13) and represented by $\boldsymbol{M}=\boldsymbol{M}(\boldsymbol{\delta},\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_l)$, and the reliable domain in the space built by the standard variables

$\boldsymbol{\delta}=[\delta_1,\delta_2,\cdots,\delta_p]$ is indicated by $\Omega_r=\{\boldsymbol{\delta}:\boldsymbol{M}(\boldsymbol{\delta},\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_l)<0\}$, then the robust reliability can be given as follows

$$\eta_r=\sup_{\boldsymbol{\delta}\in\boldsymbol{R}^p}\left\{\left\|\boldsymbol{\delta}\right\|_\infty:\boldsymbol{M}(\boldsymbol{\delta},\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_l)<0\right\}-1 \tag{14}$$

Where, $\left\|\boldsymbol{\delta}\right\|_\infty$ denote the infinity norm of the vector $\boldsymbol{\delta}=[\delta_1,\delta_2,\cdots,\delta_p]$. Essentially, the robust reliability η_r defined by (14) represents the admissible maximum degree of expansion of the uncertain parameters in the infinity topology space built by the standard interval variables under the condition of that (13) is satisfied. If $\eta_r>0$ holds, the system is reliable for all admissible uncertainties. The larger the value of η_r, the more robust the system with respect to uncertainties and the system is more reliable for this reason. So it is referred to as robust reliability in the paper as that in Ben-Haim (1996) and Guo (2007,2010).

The main objective of this chapter is to develop a method based on the robust reliability idea to deal with bounded parametric uncertainties of the system (1) and to obtain reliability-based robust fuzzy controller (6) for stabilizing the nonlinear system.

Before deriving the main results, the following lemma is given to provide a basis.

Lemma 1 (Guo, 2010). Given real matrices $\boldsymbol{Y}$, $\boldsymbol{E}_1,\boldsymbol{E}_2,\cdots,\boldsymbol{E}_n$, $\boldsymbol{F}_1,\boldsymbol{F}_2,\ldots$, and $\boldsymbol{F}_n$ with appropriate dimensions and $\boldsymbol{Y}=\boldsymbol{Y}^T$, then for any uncertain matrices $\boldsymbol{\Delta}_1=diag\{\delta_{11},\cdots,\delta_{1m_1}\}$, $\boldsymbol{\Delta}_2=diag\{\delta_{21},\cdots,\delta_{2m_2}\}$, $\cdots$, and $\boldsymbol{\Delta}_n=diag\{\delta_{n1},\cdots,\delta_{nm_n}\}$ satisfying $\left|\delta_{ij}\right|\le\alpha$ ($i=1,\ldots,n$, $j=1,\ldots,m_n$), the following inequality holds for all admissible uncertainties

$$\boldsymbol{Y}+\sum_{i=1}^{n}\left(\boldsymbol{E}_i\boldsymbol{\Delta}_i\boldsymbol{F}_i+\boldsymbol{F}_i^T\boldsymbol{\Delta}_i^T\boldsymbol{E}_i^T\right)<0 \tag{15}$$

if and only if there exist n constant positive-definite diagonal matrices $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, …, and $\boldsymbol{H}_n$ with appropriate dimensions such that

$$\boldsymbol{Y}+\sum_{i=1}^{n}\left(\boldsymbol{E}_i\boldsymbol{H}_i\boldsymbol{E}_i^T+\alpha^2\boldsymbol{F}_i^T\boldsymbol{H}_i^{-1}\boldsymbol{F}_i\right)<0 \tag{16}$$

3. Methodology and main results

3.1 Basic theory

The following commonly used Lyapunov function is considered

$$V(\boldsymbol{x}(t))=\boldsymbol{x}(t)^T\boldsymbol{P}\boldsymbol{x}(t) \tag{17}$$

where $\boldsymbol{P}$ is a symmetric positive definite matrix. The time derivative of $\boldsymbol{V}(\boldsymbol{x}(t))$ is

$$\dot{V}(\boldsymbol{x}(t))=\dot{\boldsymbol{x}}(t)^T\boldsymbol{P}\boldsymbol{x}(t)+\boldsymbol{x}(t)^T\boldsymbol{P}\dot{\boldsymbol{x}}(t) \tag{18}$$

Substituting (9) into (18), we can obtain

$$\dot{V}(\boldsymbol{x}(t))=\boldsymbol{x}^T(t)\left\{\left(\boldsymbol{A}(\boldsymbol{\rho},\mu)+\boldsymbol{B}(\boldsymbol{\rho},\mu)\boldsymbol{K}(\mu)\right)^T\boldsymbol{P}+\boldsymbol{P}\left(\boldsymbol{A}(\boldsymbol{\rho},\mu)+\boldsymbol{B}(\boldsymbol{\rho},\mu)\boldsymbol{K}(\mu)\right)\right\}\boldsymbol{x}(t) \tag{19}$$

So, $\dot{V}(\boldsymbol{x}(t))<0$ is equivalent to (20) and further equivalent to (21) that are represented as follows

$$\left(\boldsymbol{A}(\boldsymbol{\rho},\mu)+\boldsymbol{B}(\boldsymbol{\rho},\mu)\boldsymbol{K}(\mu)\right)^T\boldsymbol{P}+\boldsymbol{P}\left(\boldsymbol{A}(\boldsymbol{\rho},\mu)+\boldsymbol{B}(\boldsymbol{\rho},\mu)\boldsymbol{K}(\mu)\right)<0\ ,\ \mu\in\Omega \tag{20}$$

$$\left(\boldsymbol{A}(\boldsymbol{\rho},\mu)\boldsymbol{X}+\boldsymbol{B}(\boldsymbol{\rho},\mu)\boldsymbol{Y}(\mu)\right)^T+\left(\boldsymbol{A}(\boldsymbol{\rho},\mu)\boldsymbol{X}+\boldsymbol{B}(\boldsymbol{\rho},\mu)\boldsymbol{Y}(\mu)\right)<0\ ,\ \mu\in\Omega \tag{21}$$

In which, $\boldsymbol{X}=\boldsymbol{P}^{-1}$, $\boldsymbol{Y}(\mu)=\boldsymbol{K}(\mu)\boldsymbol{X}$ possess the following form

$$\boldsymbol{Y}(\mu)=\sum_{i=1}^{r}\mu_i(\boldsymbol{x}(t))\boldsymbol{Y}_i=\sum_{i=1}^{r}\mu_i(\boldsymbol{x}(t))\boldsymbol{K}_i\boldsymbol{X}\ \ (i=1,\ldots,r) \tag{22}$$

Let

$$\boldsymbol{Q}_{ij}(\boldsymbol{\rho},\boldsymbol{X},\boldsymbol{Y}_j)=(\boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{X}+\boldsymbol{B}_i(\boldsymbol{\rho})\boldsymbol{Y}_j)+(\boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{X}+\boldsymbol{B}_i(\boldsymbol{\rho})\boldsymbol{Y}_j)^T\ \ (i,j=1,\ldots,r) \tag{23}$$

then (21) can be written as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{Q}_{ij}(\boldsymbol{\rho},\boldsymbol{X},\boldsymbol{Y}_j)<0 \tag{24}$$

Some convex relaxations for (24) have been developed to make it tractable. Two type of relaxation are adopted here to illustrate the presented method.

3.1.1 A simple relaxation of (24) represented as follows is often used by authors (Lee, Park, & Chen 2001)

$$\boldsymbol{Q}_{ii}(\boldsymbol{\rho},\boldsymbol{X},\boldsymbol{Y}_i)<0\ ,\ \boldsymbol{Q}_{ij}(\boldsymbol{\rho},\boldsymbol{X},\boldsymbol{Y}_j)+\boldsymbol{Q}_{ji}(\boldsymbol{\rho},\boldsymbol{X},\boldsymbol{Y}_i)<0\ \ (1\le i<j\le r) \tag{25}$$

These expressions can be rewritten respectively as

$$(\boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{X}+\boldsymbol{B}_i(\boldsymbol{\rho})\boldsymbol{Y}_i)+(\boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{X}+\boldsymbol{B}_i(\boldsymbol{\rho})\boldsymbol{Y}_i)^T<0\ \ (i=1,\ldots,r) \tag{26}$$

$$\begin{aligned}(\mathbf{A}_i(\boldsymbol{\rho})\mathbf{X}+\mathbf{B}_i(\boldsymbol{\rho})\mathbf{Y}_j)+(\mathbf{A}_i(\boldsymbol{\rho})\mathbf{X}+\mathbf{B}_i(\boldsymbol{\rho})\mathbf{Y}_j)^T+(\mathbf{A}_j(\boldsymbol{\rho})\mathbf{X}+\mathbf{B}_j(\boldsymbol{\rho})\mathbf{Y}_i)+\\ +(\mathbf{A}_j(\boldsymbol{\rho})\mathbf{X}+\mathbf{B}_j(\boldsymbol{\rho})\mathbf{Y}_i)^T<0\quad(1\le i<j\le r)\end{aligned} \tag{27}$$

Expressing all the uncertain parameters $\boldsymbol{\rho}=\{\rho_1,\rho_2,\cdots,\rho_p\}$ as the standard form of (12), furthermore, the system matrices are expressed as a corresponding form of that depend on the standard interval variables $\boldsymbol{\delta}=[\delta_1,\delta_2,\cdots,\delta_p]$. Without loss of generality, suppose that all the uncertain matrices $\boldsymbol{A}_i(\boldsymbol{\rho})$ and $\boldsymbol{B}_i(\boldsymbol{\rho})$ can be expressed as

$$\boldsymbol{A}_i(\boldsymbol{\rho})=\boldsymbol{A}_{i0}+\sum_{j=1}^{p}\boldsymbol{A}_{ij}\delta_{ij}\ ,\ \boldsymbol{B}_i(\boldsymbol{\rho})=\boldsymbol{B}_{i0}+\sum_{k=1}^{q}\boldsymbol{B}_{ik}\delta_{ik}\ \ (i=1,\ldots,r) \tag{28}$$

In which, $\boldsymbol{A}_{i0}$, $\boldsymbol{B}_{i0}$, $\boldsymbol{A}_{ij}$, and $\boldsymbol{B}_{ik}$ are known real constant matrices determined by the nominal and deviational values of the basic variables. To reduce the conservatism caused by dealing with uncertainties as far as possible, representing all the matrices $\boldsymbol{A}_{ij}$ and $\boldsymbol{B}_{ik}$ as the form of the vector products as follows

$$\boldsymbol{A}_{ij} = \boldsymbol{V}_{ij1}\boldsymbol{V}_{ij2}^T,\ \boldsymbol{B}_{ik} = \boldsymbol{U}_{ij1}\boldsymbol{U}_{ik2}^T\ \ (i=1,\ldots,r,\ j=1,\ldots,p,\ k=1,\ldots,q) \tag{29}$$

In which, $\boldsymbol{V}_{ij1}$, $\boldsymbol{V}_{ij2}$, $\boldsymbol{U}_{ik1}$, and $\boldsymbol{U}_{ik2}$ are all column vectors. Denoting

$$\begin{aligned}
&\mathbf{V}_{i1} = \begin{bmatrix}\mathbf{V}_{i11} & \mathbf{V}_{i21} & \cdots & \mathbf{V}_{ip1}\end{bmatrix},\quad \mathbf{V}_{i2} = \begin{bmatrix}\mathbf{V}_{i12} & \mathbf{V}_{i22} & \cdots & \mathbf{V}_{ip2}\end{bmatrix}^T;\\
&\mathbf{U}_{i1} = \begin{bmatrix}\mathbf{U}_{i11} & \mathbf{U}_{i21} & \cdots & \mathbf{U}_{iq1}\end{bmatrix};\ \ \mathbf{U}_{i2} = \begin{bmatrix}\mathbf{U}_{i12} & \mathbf{U}_{i22} & \cdots & \mathbf{U}_{iq2}\end{bmatrix}^T,\\
&\boldsymbol{\Delta}_{i1} = diag\{\delta_{i1},\cdots,\delta_{ip}\};\ \ \boldsymbol{\Delta}_{i2} = diag\{\delta_{i1},\cdots,\delta_{iq}\};\ \ (i=1,\ldots,r)
\end{aligned} \tag{30}$$

where, the first four matrices are constructed by the column vectors involved in (29). Then, the expressions in (28) can be further written as

$$\boldsymbol{A}_i(\boldsymbol{\rho}) = \boldsymbol{A}_{i0} + \boldsymbol{V}_{i1}\boldsymbol{\Delta}_{i1}\boldsymbol{V}_{i2},\ \boldsymbol{B}_i(\boldsymbol{\rho}) = \boldsymbol{B}_{i0} + \boldsymbol{U}_{i1}\boldsymbol{\Delta}_{i2}\boldsymbol{U}_{i2}\ \ (i=1,\ldots,r) \tag{31}$$

Substituting (31) into equations (26) and (27), we can obtain

$$\begin{aligned}
&(\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_i) + (\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_i)^T + (\mathbf{V}_{i1}\boldsymbol{\Delta}_{i1}\mathbf{V}_{i2}\mathbf{X}) + (\mathbf{V}_{i1}\boldsymbol{\Delta}_{i1}\mathbf{V}_{i2}\mathbf{X})^T\\
&+(\mathbf{U}_{i1}\boldsymbol{\Delta}_{i2}\mathbf{U}_{i2}\mathbf{Y}_i) + (\mathbf{U}_{i1}\boldsymbol{\Delta}_{i2}\mathbf{U}_{i2}\mathbf{Y}_i)^T < 0\quad (i=1,\ldots,r)
\end{aligned} \tag{32}$$

$$\begin{aligned}
&(\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j) + (\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j)^T + (\mathbf{V}_{i1}\boldsymbol{\Delta}_{i1}\mathbf{V}_{i2}\mathbf{X}) + (\mathbf{V}_{i1}\boldsymbol{\Delta}_{i1}\mathbf{V}_{i2}\mathbf{X})^T\\
&\qquad + (\mathbf{U}_{i1}\boldsymbol{\Delta}_{i2}\mathbf{U}_{i2}\mathbf{Y}_j) + (\mathbf{U}_{i1}\boldsymbol{\Delta}_{i2}\mathbf{U}_{i2}\mathbf{Y}_j)^T +\\
&+(\mathbf{A}_{j0}\mathbf{X} + \mathbf{B}_{j0}\mathbf{Y}_i) + (\mathbf{A}_{j0}\mathbf{X} + \mathbf{B}_{j0}\mathbf{Y}_i)^T + (\mathbf{V}_{j1}\boldsymbol{\Delta}_{j1}\mathbf{V}_{j2}\mathbf{X}) + (\mathbf{V}_{j1}\boldsymbol{\Delta}_{j1}\mathbf{V}_{j2}\mathbf{X})^T\\
&\qquad + (\mathbf{U}_{j1}\boldsymbol{\Delta}_{j2}\mathbf{U}_{j2}\mathbf{Y}_i) + (\mathbf{U}_{j1}\boldsymbol{\Delta}_{j2}\mathbf{U}_{j2}\mathbf{Y}_i)^T < 0\quad (1 \le i < j \le r)
\end{aligned} \tag{33}$$

In terms of Lemma 1, the matrix inequality (32) holds for all admissible uncertainties if and only if there exist diagonal positive-definite matrices $\boldsymbol{E}_{i1}$ and $\boldsymbol{E}_{i2}$ with appropriate dimensions such that

$$\begin{aligned}
&(\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_i) + (\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_i)^T + \mathbf{V}_{i1}\mathbf{E}_{i1}\mathbf{V}_{i1}^T + \alpha^2(\mathbf{V}_{i2}\mathbf{X})^T\mathbf{E}_{i1}^{-1}(\mathbf{V}_{i2}\mathbf{X}) +\\
&+\mathbf{U}_{i1}\mathbf{E}_{i2}\mathbf{U}_{i1}^T + \alpha^2(\mathbf{U}_{i2}\mathbf{Y}_i)^T\mathbf{E}_{i2}^{-1}(\mathbf{U}_{i2}\mathbf{Y}_i) < 0\qquad (i=1,\ldots,r)
\end{aligned} \tag{34}$$

Similarly, (33) holds for all admissible uncertainties if and only if there exist constant diagonal positive-definite matrices $\boldsymbol{H}_{ij1}$, $\boldsymbol{H}_{ij2}$, $\boldsymbol{H}_{ij3}$, and $\boldsymbol{H}_{ij4}$ such that

$$\begin{aligned}
&(\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j) + (\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j)^T + \mathbf{V}_{i1}\mathbf{H}_{ij1}\mathbf{V}_{i1}^T + \mathbf{V}_{j1}\mathbf{H}_{ij2}\mathbf{V}_{j1}^T +\\
&+\mathbf{U}_{i1}\mathbf{H}_{ij3}\mathbf{U}_{i1}^T + \mathbf{U}_{j1}\mathbf{H}_{ij4}\mathbf{U}_{j1}^T + (\mathbf{A}_{j0}\mathbf{X} + \mathbf{B}_{j0}\mathbf{Y}_i) +\\
&(\mathbf{A}_{j0}\mathbf{X} + \mathbf{B}_{j0}\mathbf{Y}_i)^T + \alpha^2(\mathbf{V}_{i2}\mathbf{X})^T\mathbf{H}_{ij1}^{-1}(\mathbf{V}_{i2}\mathbf{X}) + \alpha^2(\mathbf{V}_{j2}\mathbf{X})^T\mathbf{H}_{ij2}^{-1}(\mathbf{V}_{j2}\mathbf{X}) +\\
&+\alpha^2(\mathbf{U}_{i2}\mathbf{Y}_j)^T\mathbf{H}_{ij3}^{-1}(\mathbf{U}_{i2}\mathbf{Y}_j) + \alpha^2(\mathbf{U}_{j2}\mathbf{Y}_i)^T\mathbf{H}_{ij4}^{-1}(\mathbf{U}_{j2}\mathbf{Y}_i) < 0\ \ (1 \le i < j \le r)
\end{aligned} \tag{35}$$

Applying the well-known Schur complement, (34) and (35) can be written respectively as

$$\begin{bmatrix}
\boldsymbol{\Xi}_i & \alpha\boldsymbol{X}\boldsymbol{V}_{i2}^T & (\alpha\boldsymbol{U}_{i2}\boldsymbol{Y}_i)^T\\
\alpha\boldsymbol{V}_{i2}\boldsymbol{X} & -\boldsymbol{E}_{i1} & \boldsymbol{0}\\
\alpha\boldsymbol{U}_{i2}\boldsymbol{Y} & \boldsymbol{0} & -\boldsymbol{E}_{i2}
\end{bmatrix} < 0\ \ (i=1,\ldots,r) \tag{36}$$

$$\begin{bmatrix} \boldsymbol{\Gamma}_{ij} & * & * & * & * \\ \alpha \boldsymbol{V}_{i2}\boldsymbol{X} & -\boldsymbol{H}_{ij1} & * & * & * \\ \alpha \boldsymbol{V}_{j2}\boldsymbol{X} & \boldsymbol{0} & -\boldsymbol{H}_{ij2} & * & * \\ \alpha \boldsymbol{U}_{i2}\boldsymbol{Y}_{j} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}_{ij3} & * \\ \alpha \boldsymbol{U}_{j2}\boldsymbol{Y}_{i} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}_{ij4} \end{bmatrix} < 0 \quad (1 \le i < j \le r) \tag{37}$$

In which,

$\boldsymbol{\Xi}_i = (\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_i) + (\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_i)^T + \boldsymbol{V}_{i1}\boldsymbol{E}_{i1}\boldsymbol{V}_{i1}^T + \boldsymbol{U}_{i1}\boldsymbol{E}_{i2}\boldsymbol{U}_{i1}^T$,

$\boldsymbol{\Gamma}_{ij} = (\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j) + (\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j)^T + (\boldsymbol{A}_{j0}\boldsymbol{X} + \boldsymbol{B}_{j0}\boldsymbol{Y}_i) + (\boldsymbol{A}_{j0}\boldsymbol{X} + \boldsymbol{B}_{j0}\boldsymbol{Y}_i)^T + \boldsymbol{V}_{i1}\boldsymbol{H}_{ij1}\boldsymbol{V}_{i1}^T +$

$\boldsymbol{V}_{j1}\boldsymbol{H}_{ij2}\boldsymbol{V}_{j1}^T + \boldsymbol{U}_{i1}\boldsymbol{H}_{ij3}\boldsymbol{U}_{i1}^T + \boldsymbol{U}_{j1}\boldsymbol{H}_{ij4}\boldsymbol{U}_{j1}^T$. "*" denotes the transposed matrices in the symmetric positions.

Consequently, the following theorem can be obtained.

Theorem 3.1. For the dynamic system (2) with the uncertain matrices represented by (31) and $|\delta_m| \le \alpha$ ($m = 1,\ldots,p$), it is asymptotically stabilizable with the state feedback controller (6) if there exist a symmetric positive-definite matrix $\boldsymbol{X}$, matrices $\boldsymbol{Y}_i$, and constant diagonal positive-definite matrices $\boldsymbol{E}_{i1}$, $\boldsymbol{E}_{i2}$, $\boldsymbol{H}_{ij1}$, $\boldsymbol{H}_{ij2}$, $\boldsymbol{H}_{ij3}$, and $\boldsymbol{H}_{ij4}$ ($1 \le i < j \le r$) such that the LMIs represented by (36) and (37) hold for all admissible uncertainties. If the feasible matrices $\boldsymbol{X}$ and $\boldsymbol{Y}_i$ are found out, then the feedback gain matrices deriving the fuzzy controller (6) can be obtained by

$$\boldsymbol{K}_i = \boldsymbol{Y}_i\boldsymbol{X}^{-1} \quad (i = 1,\ldots,r) \tag{38}$$

It should be stated that the condition of (25) is restrictive in practice. It is adopted yet here is merely to show the proposed reliability method and for comparison.

3.1.2 Some improved relaxation for (24) have also been proposed in literatures. A relaxation provided by Tuan, Apkarian, and Narikiyo (2001) is as follows

$$\boldsymbol{Q}_{ii}(\boldsymbol{\rho}, \boldsymbol{X}, \boldsymbol{Y}_i) < 0 \quad (i = 1,\ldots,r) \tag{39}$$

$$\boldsymbol{Q}_{ii}(\boldsymbol{\rho}, \boldsymbol{X}, \boldsymbol{Y}_i) + \frac{r-1}{2}\left(\boldsymbol{Q}_{ij}(\boldsymbol{\rho}, \boldsymbol{X}, \boldsymbol{Y}_j) + \boldsymbol{Q}_{ji}(\boldsymbol{\rho}, \boldsymbol{X}, \boldsymbol{Y}_i)\right) < 0 \quad (1 \le i \ne j \le r) \tag{40}$$

The expression (39) is the same completely with the first expression of (25). So, only (40) is investigated further. It can be rewritten as

$$\begin{aligned} &(\mathbf{A}_i(\boldsymbol{\rho})\mathbf{X} + \mathbf{B}_i(\boldsymbol{\rho})\mathbf{Y}_i) + (\mathbf{A}_i(\boldsymbol{\rho})\mathbf{X} + \mathbf{B}_i(\boldsymbol{\rho})\mathbf{Y}_i)^T + \\ &\frac{r-1}{2}\Big\{(\mathbf{A}_i(\boldsymbol{\rho})\mathbf{X} + \mathbf{B}_i(\boldsymbol{\rho})\mathbf{Y}_j) + (\mathbf{A}_i(\boldsymbol{\rho})\mathbf{X} + \mathbf{B}_i(\boldsymbol{\rho})\mathbf{Y}_j)^T + \\ &\qquad + (\mathbf{A}_j(\boldsymbol{\rho})\mathbf{X} + \mathbf{B}_j(\boldsymbol{\rho})\mathbf{Y}_i) + (\mathbf{A}_j(\boldsymbol{\rho})\mathbf{X} + \mathbf{B}_j(\boldsymbol{\rho})\mathbf{Y}_i)^T\Big\} < 0 \quad (1 \le i \ne j \le r) \end{aligned} \tag{41}$$

On substituting the expression (31) into (41), we obtain

$$\begin{aligned}
&\left\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)^T\right\}+\frac{r-1}{2}\left\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)^T\right\}+\\
&+\frac{r-1}{2}\left\{(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)+(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)^T\right\}+\frac{r+1}{2}\left\{(\mathbf{V}_{i1}\mathbf{\Delta}_{i1}\mathbf{V}_{i2}\mathbf{X})+(\mathbf{V}_{i1}\mathbf{\Delta}_{i1}\mathbf{V}_{i2}\mathbf{X})^T\right\}+\\
&+(\mathbf{U}_{i1}\mathbf{\Delta}_{i2}\mathbf{U}_{i2}\mathbf{Y}_i)+(\mathbf{U}_{i1}\mathbf{\Delta}_{i2}\mathbf{U}_{i2}\mathbf{Y}_i)^T+(\frac{r-1}{2}\mathbf{U}_{i1}\mathbf{\Delta}_{i2}\mathbf{U}_{i2}\mathbf{Y}_j)+(\frac{r-1}{2}\mathbf{U}_{i1}\mathbf{\Delta}_{i2}\mathbf{U}_{i2}\mathbf{Y}_j)^T\\
&+\frac{r-1}{2}\left\{(\mathbf{V}_{j1}\mathbf{\Delta}_{j1}\mathbf{V}_{j2}\mathbf{X})+(\mathbf{V}_{j1}\mathbf{\Delta}_{j1}\mathbf{V}_{j2}\mathbf{X})^T+(\mathbf{U}_{j1}\mathbf{\Delta}_{j2}\mathbf{U}_{j2}\mathbf{Y}_i)+(\mathbf{U}_{j1}\mathbf{\Delta}_{j2}\mathbf{U}_{j2}\mathbf{Y}_i)^T\right\}<0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(1\le i\ne j\le r)
\end{aligned}\tag{42}$$

In terms of Lemma 1, the matrix inequality (42) hold for all admissible uncertainties if and only if there exist constant diagonal positive-definite matrices $\boldsymbol{F}_{i1}$, $\boldsymbol{F}_{i2}$, $\boldsymbol{F}_{i3}$, $\boldsymbol{H}_{ij1}$, and $\boldsymbol{H}_{ij2}$ such that

$$\begin{aligned}
&\left\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)^T\right\}+\frac{r-1}{2}\left\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)^T\right\}\\
&+\frac{r-1}{2}\left\{(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)+(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)^T\right\}+\left(\frac{r+1}{2}\mathbf{V}_{i1}\right)\mathbf{F}_{i1}\left(\frac{r+1}{2}\mathbf{V}_{i1}\right)^T+\alpha^2(\mathbf{V}_{i2}\mathbf{X})^T\mathbf{F}_{i1}^{-1}(\mathbf{V}_{i2}\mathbf{X})\\
&+\mathbf{U}_{i1}\mathbf{F}_{i2}\mathbf{U}_{i1}^T+\alpha^2(\mathbf{U}_{i2}\mathbf{Y}_i)^T\mathbf{F}_{i2}^{-1}(\mathbf{U}_{i2}\mathbf{Y}_i)\Big\}+\left(\frac{r-1}{2}\mathbf{U}_{i1}\right)\mathbf{H}_{ij1}\left(\frac{r-1}{2}\mathbf{U}_{i1}\right)^T+\alpha^2(\mathbf{U}_{i2}\mathbf{Y}_j)^T\mathbf{H}_{ij1}^{-1}(\mathbf{U}_{i2}\mathbf{Y}_j)\\
&+\left(\frac{r-1}{2}\mathbf{V}_{j1}\right)\mathbf{F}_{j3}\left(\frac{r-1}{2}\mathbf{V}_{j1}\right)^T+\left(\frac{r-1}{2}\mathbf{U}_{j1}\right)\mathbf{H}_{ij2}\left(\frac{r-1}{2}\mathbf{U}_{j1}\right)^T+\alpha^2(\mathbf{V}_{j2}\mathbf{X})^T\mathbf{F}_{j3}^{-1}(\mathbf{V}_{j2}\mathbf{X})\\
&+\alpha^2(\mathbf{U}_{j2}\mathbf{Y}_i)^T\mathbf{H}_{ij2}^{-1}(\mathbf{U}_{j2}\mathbf{Y}_i)\Big\}<0\qquad(1\le i\ne j\le r)
\end{aligned}\tag{43}$$

Applying the Schur complement, (43) is equivalent to

$$\begin{bmatrix}
\boldsymbol{\Psi}_{ij} & * & * & * & * & * \\
\alpha\boldsymbol{V}_{i2}\boldsymbol{X} & -\boldsymbol{F}_{i1} & * & * & * & * \\
\alpha\boldsymbol{U}_{i2}\boldsymbol{Y}_i & \boldsymbol{0} & -\boldsymbol{F}_{i2} & * & * & * \\
\alpha\boldsymbol{V}_{j2}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{F}_{j3} & * & * \\
\alpha\boldsymbol{U}_{i2}\boldsymbol{Y}_j & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}_{ij1} & * \\
\alpha\boldsymbol{U}_{j2}\boldsymbol{Y}_i & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}_{ij2}
\end{bmatrix}<0\ \ (1\le i\ne j\le r)\tag{44}$$

In which, $\boldsymbol{\Psi}_{ij}=\left\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)^T\right\}+\frac{r-1}{2}\left\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)^T\right\}$

$$+\frac{r-1}{2}\left\{(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)+(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)^T\right\}+\left(\frac{r+1}{2}\mathbf{V}_{i1}\right)\mathbf{F}_{i1}\left(\frac{r+1}{2}\mathbf{V}_{i1}\right)^T+\mathbf{U}_{i1}\mathbf{F}_{i2}\mathbf{U}_{i1}^T+$$

$$\left(\frac{r-1}{2}\mathbf{V}_{j1}\right)\mathbf{F}_{j3}\left(\frac{r-1}{2}\mathbf{V}_{j1}\right)^T+\left(\frac{r-1}{2}\mathbf{U}_{i1}\right)\mathbf{H}_{ij1}\left(\frac{r-1}{2}\mathbf{U}_{i1}\right)^T+\left(\frac{r-1}{2}\mathbf{U}_{j1}\right)\mathbf{H}_{ij2}\left(\frac{r-1}{2}\mathbf{U}_{j1}\right)^T.$$

This can be summarized as follows.

Theorem 3.2. For the dynamic system (2) with the uncertain matrices represented by (31) and $|\delta_m|\le\alpha$ ($m=1,\ldots,p$), it is asymptotically stabilizable with the state feedback controller (6) if there exist a symmetric positive-definite matrix $\boldsymbol{X}$, matrices $\boldsymbol{Y}_i$, and constant diagonal

positive-definite matrices $\boldsymbol{E}_{i1}$, $\boldsymbol{E}_{i2}$, $\boldsymbol{F}_{i1}$, $\boldsymbol{F}_{i2}$, $\boldsymbol{F}_{i3}$, $\boldsymbol{H}_{ij1}$, and $\boldsymbol{H}_{ij2}$ $(1 \le i \ne j \le r)$ such that the LMIs (36) and (44) hold for all admissible uncertainties. If the feasible matrices $\boldsymbol{X}$ and $\boldsymbol{Y}_i$ are found out, the feedback gain matrices deriving the fuzzy controller (6) can then be given by (38).

3.2 Robust reliability based stabilization

In terms of Theorem 3.1, the closed-loop fuzzy system (7) is stable if all the matrix inequalities (36) and (37) hold for all admissible uncertainties. So, the performance functions used for calculation of reliability of that the uncertain system to be stable can be taken as

$$\boldsymbol{M}_i(\alpha, \boldsymbol{X}, \boldsymbol{Y}_i, \boldsymbol{E}_{i1}, \boldsymbol{E}_{i2}) = \begin{bmatrix} \boldsymbol{\Xi}_i & \alpha \boldsymbol{X} \boldsymbol{V}_{i2}^T & (\alpha \boldsymbol{U}_{i2} \boldsymbol{Y}_i)^T \\ \alpha \boldsymbol{V}_{i2} \boldsymbol{X} & -\boldsymbol{E}_{i1} & \boldsymbol{0} \\ \alpha \boldsymbol{U}_{i2} \boldsymbol{Y}_i & \boldsymbol{0} & -\boldsymbol{E}_{i2} \end{bmatrix} \quad (i = 1, \ldots, r) \tag{45}$$

$$\boldsymbol{M}_{ij}(\alpha, \boldsymbol{X}, \boldsymbol{Y}_i, \boldsymbol{Y}_j, \boldsymbol{H}_{ij1}, \boldsymbol{H}_{ij2}, \boldsymbol{H}_{ij3}, \boldsymbol{H}_{ij4}) = \begin{bmatrix} \boldsymbol{\Gamma}_{ij} & * & * & * & * \\ \alpha \boldsymbol{V}_{i2} \boldsymbol{X} & -\boldsymbol{H}_{ij1} & * & * & * \\ \alpha \boldsymbol{V}_{j2} \boldsymbol{X} & \boldsymbol{0} & -\boldsymbol{H}_{ij2} & * & * \\ \alpha \boldsymbol{U}_{i2} \boldsymbol{Y}_j & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}_{ij3} & * \\ \alpha \boldsymbol{U}_{j2} \boldsymbol{Y}_i & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}_{ij4} \end{bmatrix} \quad (1 \le i < j \le r) \tag{46}$$

in which, the expressions of $\boldsymbol{\Xi}_i$ and $\boldsymbol{\Gamma}_{ij}$ are in the same form respectively as that in (36) and (37).

Therefore, the robust reliability of the uncertain nonlinear system in the sense of stability can be expressed as

$$\eta_r = \sup_{\alpha \in \mathbf{R}^+} \left\{ \alpha : \mathbf{M}_i(\alpha, \mathbf{X}, \mathbf{Y}_i, \mathbf{E}_{i1}, \mathbf{E}_{i2}) < 0, \mathbf{M}_{ij}(\alpha, \mathbf{X}, \mathbf{Y}_i, \mathbf{Y}_j, \mathbf{H}_{ij1}, \mathbf{H}_{ij2}, \mathbf{H}_{ij3}, \mathbf{H}_{ij4}) < 0, \ 1 \le i < j \le r \right\} - 1 \tag{47}$$

where, R^+ denotes the set of all positive real numbers. The robust reliability of that the uncertain closed-loop system (7) is stable may be obtained by solving the following optimization problem

$$\begin{aligned} &\text{Maximize } \alpha \\ &\text{Subject to } \mathbf{M}_i(\alpha, \mathbf{X}, \mathbf{Y}_i, \mathbf{E}_{i1}, \mathbf{E}_{i2}) < 0 \ \ \mathbf{M}_{ij}(\alpha, \mathbf{X}, \mathbf{Y}_i, \mathbf{Y}_j, \mathbf{H}_{ij1}, \mathbf{H}_{ij2}, \mathbf{H}_{ij3}, \mathbf{H}_{ij4}) < 0 \\ &\mathbf{E}_{i1} > 0, \mathbf{E}_{i2} > 0 \ \ \mathbf{X} > 0, \mathbf{H}_{ij1} > 0, \mathbf{H}_{ij2} > 0, \mathbf{H}_{ij3} > 0, \mathbf{H}_{ij4} > 0 \quad (1 \le i < j \le r) \end{aligned} \tag{48}$$

From the viewpoint of robust stabilizing controller design, if inequalities (36) and (37) hold for all admissible uncertainties, then there exists a fuzzy controller (6) such that the closed-loop system (7) to be asymptotically stable. Therefore, the performance functions used for reliability-based design of control to stabilize the uncertain system (2) can also be taken as that of (45) and (46). So, a possible stabilizing controller satisfying the robust reliability requirement can be given by a feasible solution of the following matrix inequalities

$$\begin{aligned} &\mathbf{M}_i(\alpha^*, \mathbf{X}, \mathbf{Y}_i, \mathbf{E}_{i1}, \mathbf{E}_{i2}) < 0, \ \ \mathbf{M}_{ij}(\alpha^*, \mathbf{X}, \mathbf{Y}_i, \mathbf{Y}_j, \mathbf{H}_{ij1}, \mathbf{H}_{ij2}, \mathbf{H}_{ij3}, \mathbf{H}_{ij4}) < 0; \ \ \mathbf{E}_{i1} > 0 \mathbf{E}_{i2} > 0, \\ &\mathbf{X} > 0, \mathbf{H}_{ij1} > 0, \mathbf{H}_{ij2} > 0, \mathbf{H}_{ij3} > 0, \mathbf{H}_{ij4} > 0 \quad (1 \le i < j \le r) \quad \alpha^* = \eta_{cr} + 1 \end{aligned} \tag{49}$$

In which, $\boldsymbol{M}_i(\cdot)$ and $\boldsymbol{M}_{ij}(\cdot)$ are functions of some matrices and represented by (45) and (46) respectively. η_{cr} is the allowable robust reliability.
If the control cost is taken into account, the robust reliability based design optimization of stabilization controller can be carried out by solving the following optimization problem

$$\begin{aligned}
&\text{Minimize } \mathit{Trace}(\mathbf{N});\\
&\text{Subject to } \mathbf{M}_i(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{E}_{i1},\mathbf{E}_{i2})<0\ \mathbf{M}_{ij}(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{H}_{ij1},\mathbf{H}_{ij2},\mathbf{H}_{ij3},\mathbf{H}_{ij4})<0\\
&\mathbf{E}_{i1}>0,\ \mathbf{E}_{i2}>0,\ \mathbf{H}_{ij1}>0,\ \mathbf{H}_{ij2}>0,\ \mathbf{H}_{ij3}>0,\ \mathbf{H}_{ij4}>0,\quad (1\le i<j\le r)\\
&\begin{bmatrix}\mathbf{N} & \mathbf{I}\\ \mathbf{I} & \mathbf{X}\end{bmatrix}>0,\ \mathbf{X}>0,\mathbf{N}>0,\ \alpha^*=\eta_{cr}+1
\end{aligned} \tag{50}$$

In which, the introduced additional matrix $\boldsymbol{N}$ is symmetric positive-definite and with the same dimension as $\boldsymbol{X}$. When the feasible matrices $\boldsymbol{X}$ and $\boldsymbol{Y}_i$ are found out, the optimal fuzzy controller could be obtained by using (6) together with (38).
If Theorem 3.2 is used, the expression of $\boldsymbol{M}_{ij}(\cdot)$ corresponding to (46) becomes

$$\boldsymbol{M}_{ij}(\alpha,\boldsymbol{X},\boldsymbol{Y}_i,\boldsymbol{Y}_j,\boldsymbol{F}_{i1},\boldsymbol{F}_{i2},\boldsymbol{F}_{i3},\boldsymbol{H}_{ij1},\boldsymbol{H}_{ij2})=\begin{bmatrix}
\boldsymbol{\Psi}_{ij} & * & * & * & * & *\\
\alpha\boldsymbol{V}_{i2}\boldsymbol{X} & -\boldsymbol{F}_{i1} & * & * & * & *\\
\alpha\boldsymbol{U}_{i2}\boldsymbol{Y}_i & \boldsymbol{0} & -\boldsymbol{F}_{i2} & * & * & *\\
\alpha\boldsymbol{V}_{j2}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{F}_{j3} & * & *\\
\alpha\boldsymbol{U}_{i2}\boldsymbol{Y}_j & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}_{ij1} & *\\
\alpha\boldsymbol{U}_{j2}\boldsymbol{Y}_i & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}_{ij2}
\end{bmatrix}(1\le i\ne j\le r) \tag{51}$$

where, $\boldsymbol{\Psi}_{ij}$ is the same with that in (44). Correspondingly, (47) and (48) become respectively as follows.

$$\begin{aligned}
\eta_r=\sup_{\alpha\in\mathbf{R}^+}\{&\alpha:\mathbf{M}_i(\alpha,\mathbf{X},\mathbf{Y}_i,\mathbf{E}_{i1},\mathbf{E}_{i2})<0,\mathbf{M}_{ij}(\alpha,\mathbf{X},\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{F}_{i1},\mathbf{F}_{i2},\mathbf{F}_{i3},\mathbf{H}_{ij1},\mathbf{H}_{ij2})<0,\\
&\mathbf{E}_{i1}>0,\ \mathbf{E}_{i2}>0,\ \mathbf{X}>0,\mathbf{F}_{i1}>0,\mathbf{F}_{i2}>0,\mathbf{F}_{i3}>0,\mathbf{H}_{ij1}>0,\mathbf{H}_{ij2}>0,\ 1\le i\ne j\le r\}-1
\end{aligned} \tag{52}$$

$$\begin{aligned}
&\text{Maximize } \alpha\\
&\text{Subject to } \mathbf{M}_i(\alpha,\mathbf{X},\mathbf{Y}_i,\mathbf{E}_{i1},\mathbf{E}_{i2})<0\ \ \mathbf{M}_{ij}(\alpha,\mathbf{X},\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{F}_{i1},\mathbf{F}_{i2},\mathbf{F}_{i3},\mathbf{H}_{ij1},\mathbf{H}_{ij2})<0\\
&\mathbf{E}_{i1}>0,\mathbf{E}_{i2}>0,\mathbf{X}>0,\mathbf{F}_{i1}>0,\mathbf{F}_{i2}>0,\mathbf{F}_{i3}>0,\mathbf{H}_{ij1}>0,\mathbf{H}_{ij2}>0,\quad (1\le i\ne j\le r)
\end{aligned} \tag{53}$$

Similarly, (50) becomes

$$\begin{aligned}
&\text{Minimize } \mathit{Trace}(\mathbf{N})\\
&\text{Subject to } \mathbf{M}_i(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{E}_{i1},\mathbf{E}_{i2})<0,\ \mathbf{M}_{ij}(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{F}_{i1},\mathbf{F}_{i2},\mathbf{F}_{i3},\mathbf{H}_{ij1},\mathbf{H}_{ij2})<0\\
&\mathbf{E}_{i1}>0,\mathbf{E}_{i2}>0,\mathbf{F}_{i1}>0,\mathbf{F}_{i2}>0,\mathbf{F}_{i3}>0,\mathbf{H}_{ij1}>0,\mathbf{H}_{ij2}>0,\qquad (1\le i\ne j\le r)\\
&\begin{bmatrix}\mathbf{N} & \mathbf{I}\\ \mathbf{I} & \mathbf{X}\end{bmatrix}>0,\ \mathbf{X}>0,\mathbf{N}>0,\ \alpha^*=\eta_{cr}+1
\end{aligned} \tag{54}$$

3.3 A special case

Now, we consider a special case in which the matrices of (30) is expressed as

$$\boldsymbol{V}_{i1} = \boldsymbol{V}_1,\ \boldsymbol{V}_{i2} = \boldsymbol{V}_2,\ \boldsymbol{\Delta}_{i1} = \boldsymbol{\Delta},\ \boldsymbol{U}_{i1} = \boldsymbol{U}_{i2} = 0\ \ (i = 1,\ldots,r) \tag{55}$$

This means that the matrices $\boldsymbol{A}_i(\boldsymbol{\rho})$ in all the rules have the same uncertainty structure and the matrices $\boldsymbol{B}_i(\boldsymbol{\rho})$ become certain. In this case, (31) can be written as

$$\boldsymbol{A}_i(\boldsymbol{\rho}) = \boldsymbol{A}_{i0} + \boldsymbol{V}_1\boldsymbol{\Delta}\ \boldsymbol{V}_2,\ \boldsymbol{B}_i(\boldsymbol{\rho}) = \boldsymbol{B}_{i0}\ \ (i = 1,\ldots,r) \tag{56}$$

and the expressions involved in Theorem 3.1 can be simplified further. This is summarized in the following.

Theorem 3.3. For the dynamic system (2) with the matrices represented by (56) and $|\delta_m| \le \alpha$ ($m = 1,\ldots,p$), it is asymptotically stabilizable with the state feedback controller (6) if there exist a symmetric positive-definite matrix $\boldsymbol{X}$, matrices $\boldsymbol{Y}_i$, and constant diagonal positive-definite matrices $\boldsymbol{E}_i$ and $\boldsymbol{H}_{ij}$ ($1 \le i < j \le r$) with appropriate dimensions such that the following LMIs hold for all admissible uncertainties

$$\begin{bmatrix} \boldsymbol{\Xi}_i & \alpha\boldsymbol{X}\boldsymbol{V}_2^T \\ \alpha\boldsymbol{V}_2\boldsymbol{X} & -\boldsymbol{E}_i \end{bmatrix} < 0,\ \begin{bmatrix} \boldsymbol{\Gamma}_{ij} & \alpha\boldsymbol{X}\boldsymbol{V}_2^T \\ \alpha\boldsymbol{V}_2\boldsymbol{X} & -\boldsymbol{H}_{ij} \end{bmatrix} < 0\ \ (1 \le i < j \le r) \tag{57}$$

In which, $\boldsymbol{\Xi}_i = (\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_i) + (\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_i)^T + \boldsymbol{V}_1\boldsymbol{E}_{i1}\boldsymbol{V}_1^T$,

$\boldsymbol{\Gamma}_{ij} = (\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_j) + (\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_j)^T + (\boldsymbol{A}_{j0}\boldsymbol{X} + \boldsymbol{B}_{j0}\boldsymbol{Y}_i) + (\boldsymbol{A}_{j0}\boldsymbol{X} + \boldsymbol{B}_{j0}\boldsymbol{Y}_i)^T +$
$(2\boldsymbol{V}_1)\boldsymbol{H}_{ij}(2\boldsymbol{V}_1)^T$. If the feasible matrices $\boldsymbol{X}$ and $\boldsymbol{Y}_i$ are found out, the feedback gain matrices deriving the fuzzy controller (6) can then be given by (38).

Proof. In the case, inequalities (32) and (33) become, respectively,

$$(\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_i) + (\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_i)^T + (\boldsymbol{V}_1\boldsymbol{\Delta}\ \boldsymbol{V}_2\boldsymbol{X}) + (\boldsymbol{V}_1\boldsymbol{\Delta}\ \boldsymbol{V}_2\boldsymbol{X})^T < 0\ \ (i = 1,\ldots,r) \tag{58}$$

$$\begin{aligned} &(\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j) + (\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j)^T + 2(\mathbf{V}_1\boldsymbol{\Delta}\ \mathbf{V}_2\mathbf{X}) + 2(\mathbf{V}_1\boldsymbol{\Delta}\ \mathbf{V}_2\mathbf{X})^T \\ &+(\mathbf{A}_{j0}\mathbf{X} + \mathbf{B}_{j0}\mathbf{Y}_i) + (\mathbf{A}_{j0}\mathbf{X} + \mathbf{B}_{j0}\mathbf{Y}_i)^T < 0\quad (1 \le i < j \le r) \end{aligned} \tag{59}$$

In terms of Lemma 1, (58) holds for all admissible uncertainties if and only if there exist diagonal positive-definite matrices $\boldsymbol{E}_i$ ($i = 1,\ldots,r$) with appropriate dimensions such that

$$(\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_i) + (\boldsymbol{A}_{i0}\boldsymbol{X} + \boldsymbol{B}_{i0}\boldsymbol{Y}_i)^T + \boldsymbol{V}_1\boldsymbol{E}_i\boldsymbol{V}_1^T + \alpha^2(\boldsymbol{V}_2\boldsymbol{X})^T\boldsymbol{E}_i^{-1}(\boldsymbol{V}_2\boldsymbol{X}) < 0\ \ (i = 1,\ldots,r) \tag{60}$$

Similarly, (59) holds for all admissible uncertainties if and only if there exist constant diagonal positive-definite matrices $\boldsymbol{H}_{ij}$ such that

$$\begin{aligned} &(\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j) + (\mathbf{A}_{i0}\mathbf{X} + \mathbf{B}_{i0}\mathbf{Y}_j)^T + (\mathbf{A}_{j0}\mathbf{X} + \mathbf{B}_{j0}\mathbf{Y}_i) + \\ &+(\mathbf{A}_{j0}\mathbf{X} + \mathbf{B}_{j0}\mathbf{Y}_i)^T + (2\mathbf{V}_1)\mathbf{H}_{ij}(2\mathbf{V}_1)^T + \alpha^2(\mathbf{V}_2\mathbf{X})^T\mathbf{H}_{ij}^{-1}(\mathbf{V}_2\mathbf{X}) < 0\quad (1 \le i < j \le r) \end{aligned} \tag{61}$$

Applying Schur complement, (57) can be obtained. So, the theorem holds.

By Theorem 3.3, the performance functions used for reliability calculation can be taken as

$$\mathbf{M}_i(\alpha,\mathbf{X},\mathbf{Y}_i,\mathbf{E}_i)=\begin{bmatrix}\boldsymbol{\Xi}_i & \alpha\mathbf{X}\mathbf{V}_2^T\\ \alpha\mathbf{V}_2\mathbf{X} & -\mathbf{E}_i\end{bmatrix},\ (i=1,\ldots,r)$$
$$\mathbf{M}_{ij}(\alpha,\mathbf{X},\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{H}_{ij})=\begin{bmatrix}\boldsymbol{\Gamma}_{ij} & \alpha\mathbf{X}\mathbf{V}_2^T\\ \alpha\mathbf{V}_2\mathbf{X} & -\mathbf{H}_{ij}\end{bmatrix},\ (1\le i<j\le r) \tag{62}$$

Accordingly, a possible stabilizing controller satisfying robust reliability requirement can be obtained by a feasible solution of the following matrix inequalities

$$\mathbf{M}_i(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{E}_i)<0,\ \mathbf{M}_{ij}(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{H}_{ij})<0,\ \mathbf{X}>0,\ \mathbf{E}_i>0;\ \mathbf{H}_{ij}>0\quad(1\le i<j\le r)$$
$$\alpha^*=\eta_{cr}+1 \tag{63}$$

The optimum stabilizing controller based on the robust reliability and control cost can be obtained by solving the following optimization problem

$$\begin{aligned}&\text{Minimize } \mathit{Trace}(\mathbf{N})\\ &\text{Subject to } \mathbf{M}_i(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{E}_i)<0,\ \mathbf{M}_{ij}(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{H}_{ij})<0;\ \mathbf{E}_i>0,\mathbf{H}_{ij}>0,\ (1\le i<j\le r)\\ &\begin{bmatrix}\mathbf{N} & \mathbf{I}\\ \mathbf{I} & \mathbf{X}\end{bmatrix}>0,\ \mathbf{X}>0,\mathbf{N}>0,\ \alpha^*=\eta_{cr}+1\end{aligned} \tag{64}$$

Similarly, the expressions involved in Theorem 3.2 can also be simplified and the corresponding result is summarized in the following.

Theorem 3.4. For the dynamic system (2) with the matrices represented by (56) and $|\delta_m|\le\alpha$ ($m=1,\ldots,p$), it is asymptotically stabilizable with the state feedback controller (6) if there exist a symmetric positive-definite matrix $\boldsymbol{X}$, matrices $\boldsymbol{Y}_i$, and constant diagonal positive-definite matrices $\boldsymbol{E}_i$ and $\boldsymbol{H}_{ij}$ ($1\le i\ne j\le r$) with appropriate dimensions such that the following LMIs hold for all admissible uncertainties

$$\begin{bmatrix}\boldsymbol{\Xi}_i & \alpha\boldsymbol{X}\boldsymbol{V}_2^T\\ \alpha\boldsymbol{V}_2\boldsymbol{X} & -\boldsymbol{E}_i\end{bmatrix}<0,\ \begin{bmatrix}\boldsymbol{\Psi}_{ij} & (\alpha\boldsymbol{V}_2\boldsymbol{X})^T\\ (\alpha\boldsymbol{V}_2\boldsymbol{X}) & -\boldsymbol{H}_{ij}\end{bmatrix}<0\ (1\le i\ne j\le r) \tag{65}$$

In which, $\boldsymbol{\Xi}_i=(\boldsymbol{A}_{i0}\boldsymbol{X}+\boldsymbol{B}_{i0}\boldsymbol{Y}_i)+(\boldsymbol{A}_{i0}\boldsymbol{X}+\boldsymbol{B}_{i0}\boldsymbol{Y}_i)^T+\boldsymbol{V}_{i1}\boldsymbol{E}_{i1}\boldsymbol{V}_{i1}^T$,

$\boldsymbol{\Psi}_{ij}=(r\boldsymbol{V}_1)\boldsymbol{H}_{ij}(r\boldsymbol{V}_1)^T+\{(\boldsymbol{A}_{i0}\boldsymbol{X}+\boldsymbol{B}_{i0}\boldsymbol{Y}_i)+(\boldsymbol{A}_{i0}\boldsymbol{X}+\boldsymbol{B}_{i0}\boldsymbol{Y}_i)^T\}$

$+\frac{r-1}{2}\{(\boldsymbol{A}_{i0}\boldsymbol{X}+\boldsymbol{B}_{i0}\boldsymbol{Y}_j)+(\boldsymbol{A}_{i0}\boldsymbol{X}+\boldsymbol{B}_{i0}\boldsymbol{Y}_j)^T+(\boldsymbol{A}_{j0}\boldsymbol{X}+\boldsymbol{B}_{j0}\boldsymbol{Y}_i)+(\boldsymbol{A}_{j0}\boldsymbol{X}+\boldsymbol{B}_{j0}\boldsymbol{Y}_i)^T\}$. If the feasible matrices X and Y_i are found out, the feedback gain matrices deriving the fuzzy controller (6) can then be obtained by (38).

Proof. (42) can be rewritten as

$$\begin{aligned}&\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)^T\}+\frac{r-1}{2}\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)^T\\ &+(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)+(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)^T\}+r\{(\mathbf{V}_1\boldsymbol{\Delta}\ \mathbf{V}_2\mathbf{X})+(\mathbf{V}_1\boldsymbol{\Delta}\ \mathbf{V}_2\mathbf{X})^T\}<0\\ &\qquad\qquad(1\le i\ne j\le r)\end{aligned} \tag{66}$$

In terms of Lemma 1, (66) holds for all admissible uncertainties if and only if there exist diagonal positive-definite matrices $\boldsymbol{H}_{ij}$ such that

$$\begin{aligned}&\left\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_i)^T\right\}+\frac{r-1}{2}\left\{(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)+(\mathbf{A}_{i0}\mathbf{X}+\mathbf{B}_{i0}\mathbf{Y}_j)^T+(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)\right.\\&\left.+(\mathbf{A}_{j0}\mathbf{X}+\mathbf{B}_{j0}\mathbf{Y}_i)^T\right\}+(r\mathbf{V}_1)\mathbf{H}_{ij}(r\mathbf{V}_1)^T+\alpha^2\left(\mathbf{V}_2\mathbf{X}\right)^T\mathbf{H}_{ij}^{-1}\left(\mathbf{V}_2\mathbf{X}\right)<0\quad 1\le i\ne j\le r\end{aligned} \tag{67}$$

Applying Schur complement, (67) is equivalent to the second expression of (65). So, the theorem holds.

By Theorem 3.4, the performance functions used for reliability calculation can be taken as

$$\begin{aligned}\mathbf{M}_i(\alpha,\mathbf{X},\mathbf{Y}_i,\mathbf{E}_i)&=\begin{bmatrix}\boldsymbol{\Xi}_i & \left(\alpha\mathbf{V}_2\mathbf{X}\right)^T\\ \left(\alpha\mathbf{V}_2\mathbf{X}\right) & -\mathbf{E}_i\end{bmatrix},\quad (i=1,\ldots,r)\\ \mathbf{M}_{ij}(\alpha,\mathbf{X},\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{H}_{ij})&=\begin{bmatrix}\boldsymbol{\Psi}_{ij} & \left(\alpha\mathbf{V}_2\mathbf{X}\right)^T\\ \left(\alpha\mathbf{V}_2\mathbf{X}\right) & -\mathbf{H}_{ij}\end{bmatrix},\quad (1\le i\ne j\le r)\end{aligned} \tag{68}$$

So, design of the optimal controller based on the robust reliability and control cost could be carried out by solving the following optimization problem

$$\begin{aligned}&\text{Minimize } \mathit{Trace}(\mathbf{N})\\&\text{Subject to } \mathbf{M}_i(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{E}_i)<0,\ \mathbf{M}_{ij}(\alpha^*,\mathbf{X},\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{H}_{ij})<0,\ \mathbf{E}_i>0,\mathbf{H}_{ij}>0\ \ (1\le i\ne j\le r)\\&\begin{bmatrix}\mathbf{N} & \mathbf{I}\\ \mathbf{I} & \mathbf{X}\end{bmatrix}>0,\ \ \mathbf{X}>0,\mathbf{N}>0\ \ (\alpha^*=\eta_{cr}+1)\end{aligned} \tag{69}$$

4. Numerical examples

Example 1. Consider a simple uncertain nonlinear mass-spring-damper mechanical system with the following dynamic equation (Tanaka, Ikeda, & Wang 1996)

$$\ddot{x}(t)+\dot{x}(t)+c(t)x(t)=(1+0.13\dot{x}^3(t))u(t)$$

Where $c(t)$ is the uncertain term satisfying $c(t)\in[0.5,1.81]$.

Assume that $x(t)\in[-1.5,1.5]$, $\dot{x}(t)\in[-1.5,1.5]$. Using the following fuzzy sets

$$F_1(\dot{x}(t))=0.5+\frac{\dot{x}^3(t)}{6.75},\ \ F_2(\dot{x}(t))=0.5-\frac{\dot{x}^3(t)}{6.75}$$

The uncertain nonlinear system can be represented by the following fuzzy model

Plant Rule 1: IF $\dot{x}(t)$ is about F_1, THEN $\dot{\boldsymbol{x}}(t)=\boldsymbol{A}_1\boldsymbol{x}(t)+\boldsymbol{B}_1\boldsymbol{u}(t)$

Plant Rule 2: IF $\dot{x}(t)$ is about F_2, THEN $\dot{\boldsymbol{x}}(t)=\boldsymbol{A}_2\boldsymbol{x}(t)+\boldsymbol{B}_2\boldsymbol{u}(t)$

Where

$$\mathbf{x}(t)=\begin{bmatrix}\dot{x}(t)\\ x(t)\end{bmatrix},\ \mathbf{A}_1=\mathbf{A}_2=\begin{bmatrix}-1 & -c\\ 1 & 0\end{bmatrix},\ \mathbf{B}_1=\begin{bmatrix}1.43875\\ 0\end{bmatrix},\ \mathbf{B}_2=\begin{bmatrix}0.56125\\ 0\end{bmatrix}$$

Expressing the uncertain parameter c as the normalized form, $c = 1.155 + 0.655\delta$, furthermore, the system matrices are expressed as

$$\boldsymbol{A}_1 = \boldsymbol{A}_{10} + \boldsymbol{V}_1 \Delta \boldsymbol{V}_2,\ \boldsymbol{A}_2 = \boldsymbol{A}_{20} + \boldsymbol{V}_1 \Delta \boldsymbol{V}_2,\ \boldsymbol{B}_1 = \boldsymbol{B}_{10},\ \boldsymbol{B}_2 = \boldsymbol{B}_{20}.$$

In which

$$\boldsymbol{A}_{10} = \boldsymbol{A}_{20} = \begin{bmatrix} -1 & -1.155 \\ 1 & 0 \end{bmatrix}, \boldsymbol{V}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \boldsymbol{V}_2 = \begin{bmatrix} 0 & -0.655 \end{bmatrix},\ \Delta = \delta.$$

By solving the optimization problem of (69) with $\alpha^* = 1$ and $\alpha^* = 3$ respectively, the gain matrices are obtained as follows

$$\boldsymbol{K}_1 = \begin{bmatrix} -0.0567 & -0.1446 \end{bmatrix},\ \boldsymbol{K}_2 = \begin{bmatrix} -0.0730 & -0.1768 \end{bmatrix}\ (\alpha^* = 1);$$

$$\boldsymbol{K}_1 = \begin{bmatrix} -0.3645 & -1.0570 \end{bmatrix},\ \boldsymbol{K}_2 = \begin{bmatrix} -0.9191 & -2.4978 \end{bmatrix}\ (\alpha^* = 3).$$

When the initial value of the state is taken as $\boldsymbol{x}(0) = \begin{bmatrix} -1 & -1.3 \end{bmatrix}^T$, the simulation results of the controlled system with the uncertain parameter generated randomly within the allowable range $c(t) \in [0.5, 1.81]$ are shown in Fig. 1.

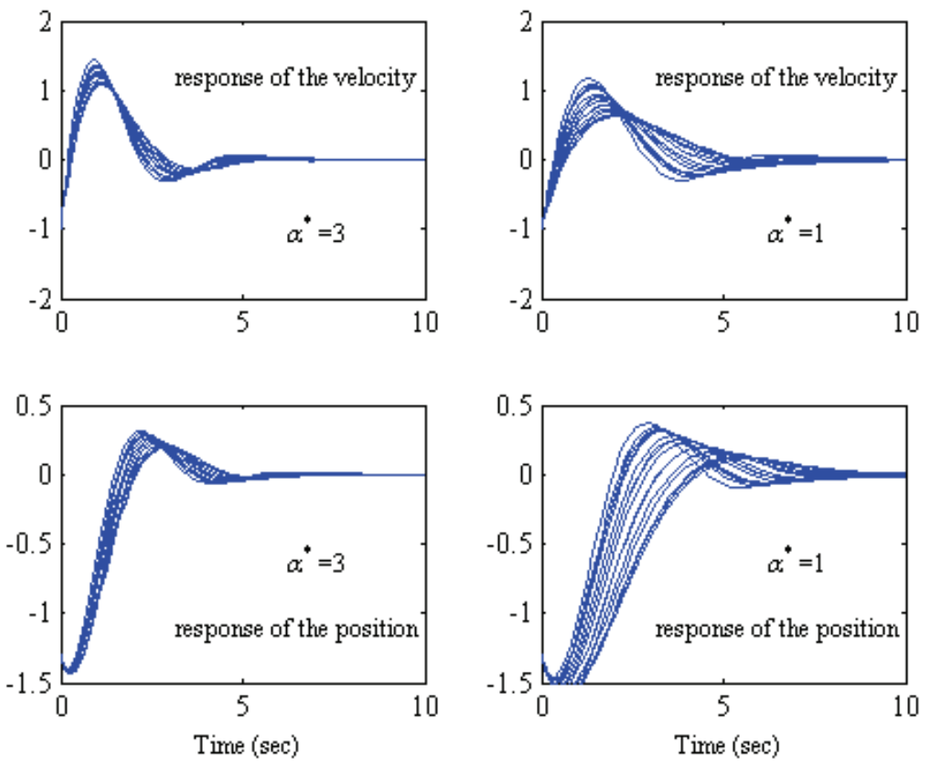


Fig. 1. Simulation of state trajectories of the controlled system (The uncertain parameter c is generated randomly within [0.5, 1.81])

Example 2. Consider the problem of stabilizing the chaotic Lorenz system with parametric uncertainties as follows (Lee, Park, & Chen, 2001)

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}-\sigma x_1(t)+\sigma x_2(t)\\ rx_1(t)-x_2(t)-x_1(t)x_3(t)\\ x_1(t)x_2(t)-bx_3(t)\end{bmatrix}$$

For the purpose of comparison, the T-S fuzzy model of the chaotic Lorenz system is constructed as

Plant Rule 1: IF $x_1(t)$ is about M_1 THEN $\dot{\mathbf{x}}(t)=\mathbf{A}_1\mathbf{x}(t)+\mathbf{B}_1\mathbf{u}(t)$
Plant Rule 2: IF $x_1(t)$ is about M_2 THEN $\dot{\mathbf{x}}(t)=\mathbf{A}_2\mathbf{x}(t)+\mathbf{B}_2\mathbf{u}(t)$

Where

$$\mathbf{A}_1=\begin{bmatrix}-\sigma & \sigma & 0\\ r & -1 & -M_1\\ 0 & M_1 & -b\end{bmatrix},\ \mathbf{A}_2=\begin{bmatrix}-\sigma & \sigma & 0\\ r & -1 & -M_2\\ 0 & M_2 & -b\end{bmatrix}$$

The input matrices $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, and the membership functions are respectively

$$\mathbf{B}_1=\mathbf{B}_2=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T,\ \mu_1(\mathbf{x}(t))=\frac{-x_1(t)+M_2}{M_2-M_1},\ \mu_2(\mathbf{x}(t))=\frac{x_1(t)-M_1}{M_2-M_1}.$$

The nominal values of (σ,r,b) are (10, 28, 8/3), and choosing $[M_1,M_2]=[-20,30]$. All system parameters are uncertain-but-bounded within 30% of their nominal values.
The gain matrices for deriving the stabilizing controller (6) given by Lee, Park, and Chen (2001) are

$$\boldsymbol{K}_{1L}=\begin{bmatrix}-295.7653 & -137.2603 & -8.0866\end{bmatrix},\ \boldsymbol{K}_{2L}=\begin{bmatrix}-443.0647 & -204.8089 & 12.6930\end{bmatrix}.$$

(1) Reliability-based feasible solutions

In order to apply the presented method, all the uncertain parameters (σ,r,b) are expressed as the following normalized form

$$\sigma=10+3\delta_1,\ r=28+8.4\delta_2,\ b=8/3+0.8\delta_3.$$

Furthermore, the system matrices can be expressed as

$$\boldsymbol{A}_1=\boldsymbol{A}_{10}+\boldsymbol{V}_1\boldsymbol{\Delta}\boldsymbol{V}_2,\ \boldsymbol{A}_2=\boldsymbol{A}_{20}+\boldsymbol{V}_1\boldsymbol{\Delta}\boldsymbol{V}_2,\ \boldsymbol{B}_1=\boldsymbol{B}_{10},\ \boldsymbol{B}_2=\boldsymbol{B}_{20}.$$

In which

$$\mathbf{A}_{10}=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & 20\\ 0 & -20 & -8/3\end{bmatrix},\ \mathbf{A}_{20}=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & -30\\ 0 & 30 & -8/3\end{bmatrix},\ \mathbf{V}_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\ \mathbf{V}_2=\begin{bmatrix}-3 & 3 & 0\\ 8.4 & 0 & 0\\ 0 & 0 & -0.8\end{bmatrix},$$

$$\boldsymbol{\Delta}=diag\{\delta_1,\delta_2,\delta_3\},\ \mathbf{B}_{10}=\mathbf{B}_{20}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$$

By solving the matrix inequalities corresponding to (63) with $\alpha^*=1$, the gain matrices are found to be

$$\mathbf{K}_1=\begin{bmatrix}-84.2940 & -23.7152 & -2.4514\end{bmatrix},\ \mathbf{K}_2=\begin{bmatrix}-84.4669 & -23.6170 & 3.8484\end{bmatrix}$$

The common positive definite matrix $\boldsymbol{X}$ and other feasible matrices obtained are as follows

$$\boldsymbol{X} = \begin{bmatrix} 0.0461 & -0.0845 & 0.0009 \\ -0.0845 & 0.6627 & -0.0027 \\ 0.0009 & -0.0027 & 0.6685 \end{bmatrix},\ \boldsymbol{E}_1 = diag\{3.4260, 2.3716, 1.8813\},$$

$$\boldsymbol{E}_2 = diag\{3.4483, 2.2766, 1.9972\},\ \boldsymbol{H} = diag\{2.6535, 1.9734, 1.3318\}.$$

Again, by solving the matrix inequalities corresponding to (63) with $\alpha^* = 2$, which means that the allowable variation of all the uncertain parameters are within 60% of their nominal values, we obtain

$$\boldsymbol{K}_1 = \begin{bmatrix} -123.6352 & -42.4674 & -4.4747 \end{bmatrix},\ \boldsymbol{K}_2 = \begin{bmatrix} -125.9081 & -42.8129 & 6.8254 \end{bmatrix},$$

$$\boldsymbol{X} = \begin{bmatrix} 1.0410 & -1.7704 & 0.0115 \\ -1.7704 & 7.7159 & -0.0360 \\ 0.0115 & -0.0360 & 7.7771 \end{bmatrix},\ \boldsymbol{E}_1 = diag\{98.7271, 44.0157, 22.7070\},$$

$$\boldsymbol{E}_2 = diag\{101.9235, 42.7451, 24.7517\},\ \boldsymbol{H} = diag\{68.8833, 31.0026, 13.8173\}.$$

Clearly, the control inputs of the controllers obtained in the paper in the two cases are all lower than that of Lee, Park, and Chen (2001).

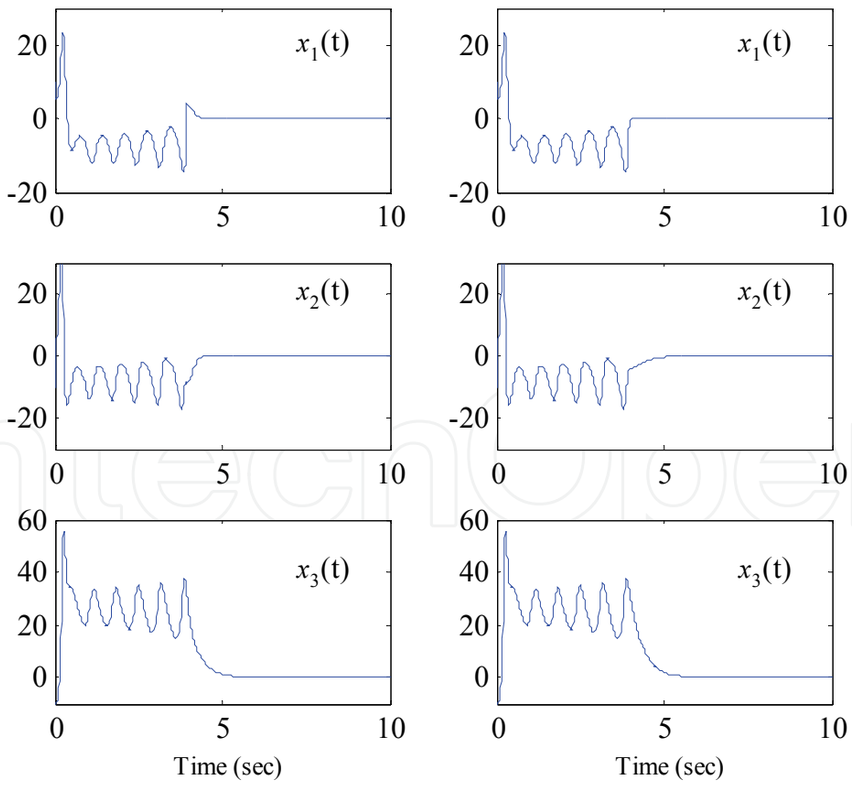


Fig. 2. State trajectories of the controlled nominal chaotic Lorenz system (On the left- and right-hand sides are results respectively of the controller of Lee, Park, and Chen (2001) and of the controller obtained in this paper)

(2) Robust reliability based design of optimal controller

Firstly, if Theorem 3.3 is used, by solving a optimization problem corresponding to (64) with $\alpha^* = 1$, the gain matrices as follows for deriving the controller are obtained

$$\boldsymbol{K}_{1G} = \begin{bmatrix} -20.8512 & -13.5211 & -3.2536 \end{bmatrix},\ \boldsymbol{K}_{2G} = \begin{bmatrix} -21.2143 & -13.1299 & 4.3799 \end{bmatrix}.$$

The norm of the gain matrices are respectively $\|\boldsymbol{K}_{1G}\| = 25.0635$ and $\|\boldsymbol{K}_{2G}\| = 25.3303$. So, there exist relations

$$\|\boldsymbol{K}_{1L}\| = 326.1639 = 13.0135\|\boldsymbol{K}_{1G}\|,\ \|\boldsymbol{K}_{2L}\| = 488.2767 = 19.2764\|\boldsymbol{K}_{2G}\|.$$

To examine the effect of the controllers, the initial values of the states of the Lorenz system are taken as $\boldsymbol{x}(0) = \begin{bmatrix} 10 & -10 & -10 \end{bmatrix}^T$, the control input is activated at *t*=3.89s, all as that of Lee, Park, and Chen (2001), the simulated state trajectories of the controlled Lorenz system without uncertainty are shown in Fig. 2. In which, on the left- and right-hand sides are results of the controller of Lee, Park, and Chen (2001) and of the controller obtained in this paper respectively. Simulations of the corresponding control inputs are shown in Fig. 3, in which, the dash-dot line and the solid line represent respectively the input of the controller of Lee, Park, and Chen (2001) and of the controller in the paper.

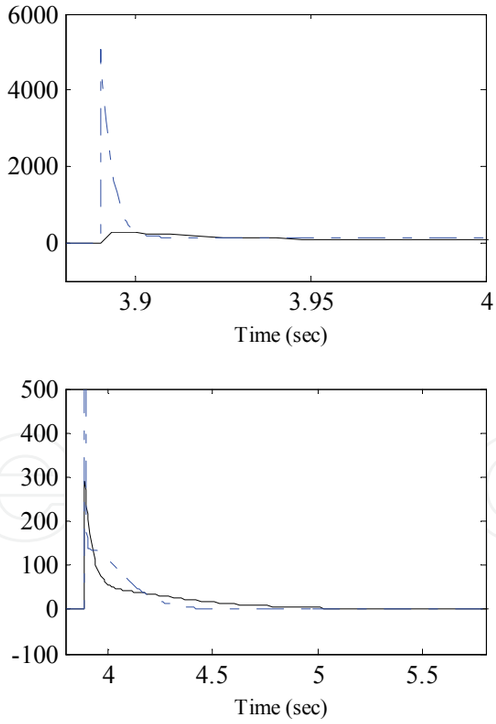


Fig. 3. Control input of the two controllers (dash-dot line and solid line represent respectively the result of Lee, Park, and Chen (2001) and the result of the paper)

The simulated state trajectories and phase trajectory of the controlled Lorenz system are shown respectively in Figs. 4 and 5, in which, all the uncertain parameters are generated randomly within the allowable ranges.

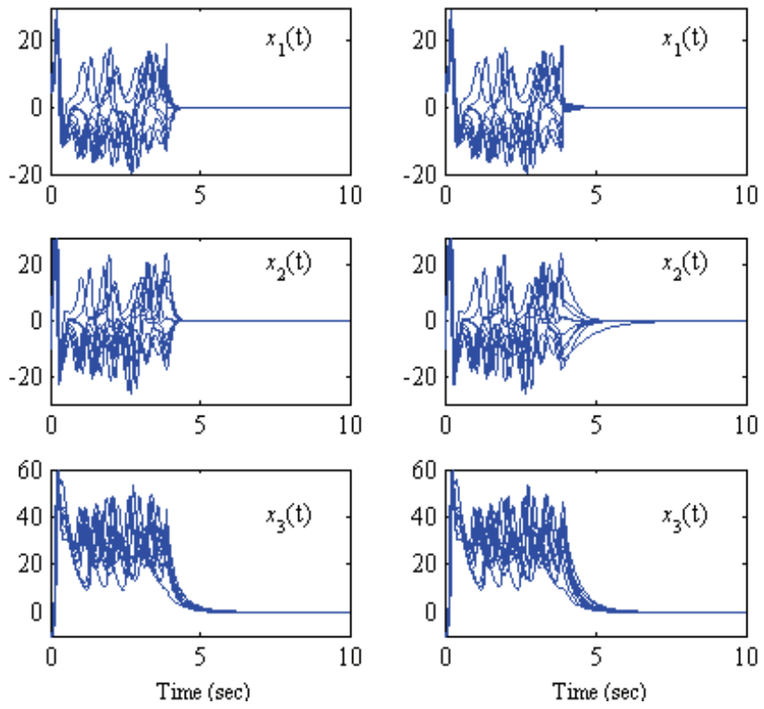


Fig. 4. Ten-times simulated state trajectories of the controlled chaotic Lorenz system with parametric uncertainties (all uncertain parameters are generated randomly within the allowable ranges, and on the left- and right-hand sides are respectively the results of controllers in Lee, Park, and Chen (2001) and in the paper)

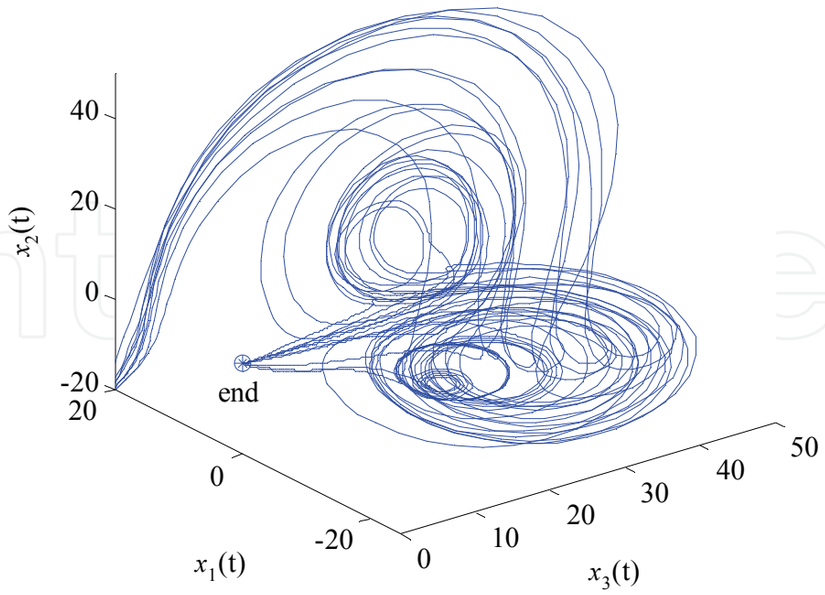


Fig. 5. Ten-times simulated phase trajectories of the parametric uncertain Lorenz system controlled by the presented method (all parameters are generated randomly within their allowable ranges)

It can be seen that the controller obtained by the presented method is effective, and the control effect has no evident difference with that of the controller in Lee, Park, and Chen (2001), but the control input of it is much lower. This shows that the presented method is much less conservative.

Taking $\alpha = 3$, which means that the allowable variation of all the uncertain parameters are within 90% of their nominal values, by applying Theorem 3.3 and solving a corresponding optimization problem of (64) with $\alpha^* = 3$, the gain matrices for deriving the fuzzy controller obtained by the presented method become

$$\mathbf{K}_{1G} = \begin{bmatrix} -54.0211 & -32.5959 & -6.5886 \end{bmatrix}, \ \mathbf{K}_{2G} = \begin{bmatrix} -50.0340 & -30.6071 & 10.4215 \end{bmatrix}.$$

Obviously, the input of the controller in this case is also much lower than that of the controller obtained by Lee, Park, and Chen (2001).

Secondly, when Theorem 3.4 is used, by solving two optimization problems corresponding to (69) with $\alpha^* = 1$ and $\alpha^* = 3$ respectively, the gain matrices for deriving the controller are found to be

$$\mathbf{K}_{1G} = \begin{bmatrix} -20.8198 & -13.5543 & -3.2560 \end{bmatrix}, \ \mathbf{K}_{2G} = \begin{bmatrix} -21.1621 & -13.1451 & 4.3928 \end{bmatrix} \ (\alpha^* = 1).$$

$$\mathbf{K}_{1G} = \begin{bmatrix} -54.0517 & -32.6216 & -6.6078 \end{bmatrix}, \ \mathbf{K}_{2G} = \begin{bmatrix} -50.0276 & -30.6484 & 10.4362 \end{bmatrix} \ (\alpha^* = 3)$$

Note that the results based on Theorem 3.4 are in agreement, approximately, with those based on Theorem 3.3.

5. Conclusion

In this chapter, stability of parametric uncertain nonlinear systems was studied from a new point of view. A robust reliability procedure was presented to deal with bounded parametric uncertainties involved in fuzzy control of nonlinear systems. In the method, the T-S fuzzy model was adopted for fuzzy modeling of nonlinear systems, and the parallel-distributed compensation (PDC) approach was used to control design. The stabilizing controller design of uncertain nonlinear systems were carried out by solving a set of linear matrix inequalities (LMIs) subjected to robust reliability for feasible solutions, or by solving a robust reliability based optimization problem to obtain optimal controller. In the optimal controller design, both the robustness with respect to uncertainties and control cost can be taken into account simultaneously. Formulations used for analysis and synthesis are within the framework of LMIs and thus can be carried out conveniently. It is demonstrated, via numerical simulations of control of a simple mechanical system and of the chaotic Lorenz system, that the presented method is much less conservative and is effective and feasible. Moreover, the bounds of uncertain parameters are not required strictly in the presented method. So, it is applicable for both the cases that the bounds of uncertain parameters are known and unknown.

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The main objective of this monograph is to present a broad range of well worked out, recent theoretical and application studies in the field of robust control system analysis and design. The contributions presented here include but are not limited to robust PID, H-infinity, sliding mode, fault tolerant, fuzzy and QFT based control systems. They advance the current progress in the field, and motivate and encourage new ideas and solutions in the robust control area.

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