

# We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

186,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index  
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?  
Contact [book.department@intechopen.com](mailto:book.department@intechopen.com)

Numbers displayed above are based on latest data collected.  
For more information visit [www.intechopen.com](http://www.intechopen.com)



# Radiative Transfer Theory for Layered Random Media

Saba Mudaliar

*Sensors Directorate, Air Force Research Laboratory, Hanscom AFB  
USA*

## 1. Introduction

When a random medium is sparse and the extent or size of the random medium is small, then a single scattering theory is sufficient; multiple scattering effects are negligible (Tatarskii, 1971; Ishimaru, 1997; Tsang et al., 1985). However, when the medium is not sparse or when the extent of the scattering medium is large, then multiple scattering becomes important. In principle, one can use the wave equations or Maxwell's equations to carry out a multiple scattering analysis (Foldy, 1945; Lax, 1951; Twersky, 1980). This procedure, also known as the statistical wave approach, is quite rigorous and takes into consideration all multiple scattering processes involved in the problem. However, the methods of analysis and solution techniques are rather complicated. One is forced to impose various approximations in order to perform numerical computations and arrive at useful results. On the other hand, the radiative transfer theory (RTT), another approach to this problem, is conceptually simple and at the same time very efficient in modelling multiple scattering processes. Furthermore, there are well-established techniques for numerical analysis of the radiative transfer equations (Clough et al., 2005; Stamnes et al., 1988; Berk et al., 1998; Lenoble, 1985).

However, the RTT is heuristic and lacks the rigour of the statistical wave theory. The fundamental quantity in the RTT is the specific intensity, which is a measure of energy flux density per unit area, per unit steradian. Although the concept of specific intensity has many desirable properties, the fact that RTT deals entirely with intensities means that it does not possess phase information and it cannot adequately describe wave phenomena such as diffraction and interference. The basic equation of the RTT is the radiative transfer equation, given as (Chandrasekar, 1960; Sobolev, 1963; Ishimaru, 1997)

$$\hat{s} \cdot \nabla I(\mathbf{r}, \hat{s}) + \gamma I(\mathbf{r}, \hat{s}) = \int P(\hat{s}, \hat{s}') I(\mathbf{r}, \hat{s}') d\Omega', \quad (1)$$

where  $I$  is the radiant intensity, which is a phase-space quantity at position  $\mathbf{r}$  and direction  $\hat{s}$ ;  $\gamma$  is the extinction coefficient, which is a measure of loss of energy in direction  $\hat{s}$  due to scattering in other directions.  $P$  is the phase function, representing the increase in energy density in direction  $\hat{s}$  due to scattering from neighbouring elements.  $d\Omega'$  is the solid angle element subtended by the radiant intensity in direction  $\hat{s}$ . Equation (1) is the radiative transfer equation, which may be regarded as a statement of conservation of radiant intensity. This scalar transport equation is inappropriate when the scattering medium has anisotropic fluctuations or if it involves boundaries. Even for models with spherical scatterers the scalar approach is inaccurate (Kattawar & Adams, 1990; Stamnes, 1994; Hasekamp et al.,

2002; Stam & Hovenier, 2005; Levy et al., 2004; Mishchenko et al., 2006). It is important in such situations to use the following vector version of the transport equation

$$\hat{s} \cdot \nabla \mathbf{I}(\mathbf{r}, \hat{s}) + \bar{\gamma} \mathbf{I}(\mathbf{r}, \hat{s}) = \int \bar{\mathbf{P}}(\hat{s}, \hat{s}') \mathbf{I}(\mathbf{r}, \hat{s}') d\Omega'. \quad (2)$$

where  $\mathbf{I}$  is the Stokes vector,  $\bar{\gamma}$  is the extinction matrix, and  $\bar{\mathbf{P}}$  is the phase matrix. Quite often in applications these quantities are modelled using empirical data. One may also calculate these quantities (Tsang et al., 1985; Ulaby et al., 1986) using wave scattering theory if one knows the statistical characteristics of the medium.

Most problems encountered in applications involve boundaries. Hence the radiative transfer (RT) equations must be supplemented by boundary conditions. Among the very early applications of the RTT, the plane parallel geometry has been thoroughly studied (Chandrasekar, 1960). However, in those applications (e.g., atmosphere) the boundaries are nonscattering and hence do not significantly impact the scattering process. There are, indeed, several other applications such as subsurface sensing (Moghaddam et al., 2007), optical mirrors (Amra, 1994; Elson, 1995), and seismology (Sato & Fehler, 1998) where the boundaries do scatter, thereby influencing the multiple scattering process.

Consider the problem of two scattering media separated by a boundary. The geometry of this problem is shown in Figure 1. Scattering media 1 and 2 are separated by a boundary  $\Sigma$ . The permittivities of the media have a deterministic part  $\epsilon_j$  and a randomly fluctuating part  $\tilde{\epsilon}_j$ . Note that there is an index mismatch between the background permittivities of medium 1 and medium 2. Thus the boundary is, indeed, a scattering boundary. Let  $I_1$  and  $I_2$  be the radiant intensities in medium 1 and medium 2, respectively. We use the superscript “in” to denote that part of the radiant intensity that goes towards the boundary and the superscript “out” to denote the part of radiant intensity that goes away from the boundary. The boundary conditions used for this kind of problem are

$$I_1^{out} = \mathcal{R}_{21} I_1^{in} + \mathcal{T}_{12} I_2^{in} \quad (3a)$$

$$I_2^{out} = \mathcal{R}_{12} I_2^{in} + \mathcal{T}_{21} I_1^{in} \quad (3b)$$

where  $\mathcal{R}$  and  $\mathcal{T}$  symbolically represent the reflection and transmission processes that take place at the boundary. The first subscript indicates the region where the scattered beam travels. The second subscript indicates the region where the incident beam originates. For instance,  $\mathcal{R}_{12}$  represents the reflection at the boundary for a beam incident from below. Note that these boundary conditions are based on energy conservation at the boundary. For bounded geometries the system of equations that needs to be solved comprises the RT equation (1) along with the equation associated with boundary conditions (3).

One should point out that the RTT as applied to a particular problem is a model constructed on certain hypotheses and assumptions. In most papers on applications using the RTT the conditions and assumptions involved are rarely stated or discussed. Since energy balance considerations are employed in constructing the RT equation people often take it as a fundamental axiom that requires no further explanation or justification. Even in a few works where the underlying assumptions are mentioned the particular approximations involved are described in terms of special technical terminologies specific to the discipline where it is used. One good way to understand in more general terms the RT approach and its underlying assumptions is to connect it with the more rigorous statistical wave approach. For the case of an unbounded random medium this kind of study was carried out in the 1970s

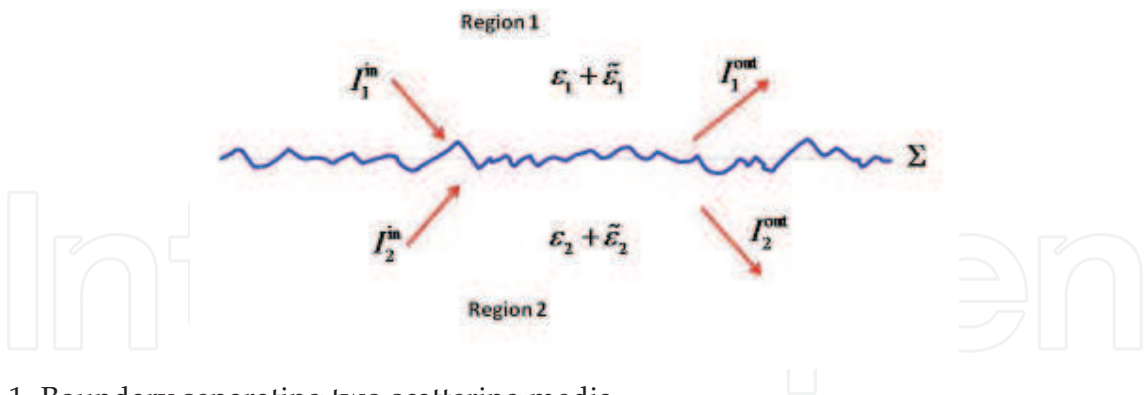


Fig. 1. Boundary separating two scattering media.

(Barabanenkov et al., 1972). From that study we learn that the radiative transfer theory can be applied under the following conditions:

- 1. Quasi-stationary field approximation.
- 2. Weak fluctuations (first-order approximation to Mass and Intensity operators)
- 3. Statistical homogeneity of the medium fluctuations.

However, our problem has bounded structures which may be planar or randomly rough. Therefore it remains to be seen whether the conditions arrived at in the case of unbounded random media will be sufficient for our problem.

In this work we employ a statistical wave approach using surface scattering operators (Voronovich, 1999; Mudaliar, 2005) to derive the coherence functions, and hence make a transition (using Wigner transforms) to transport equations for our multilayer problem. In this process we find that there are more conditions implied when we choose to apply the RT approach to our problem than it is widely believed to be necessary. One such condition is the weak surface correlation approximation. This means that the RT approach places certain restrictions on the type of rough interfaces that it can model accurately.

This chapter is organized as follows. In Section 2 we consider layered random media with planar boundaries. In Section 3 we consider the corresponding problem with rough interfaces. This chapter concludes in Section 4 with a summary and a discussion of our main findings.

2. Layered random medium with planar interfaces

Multiple scattering in layered scattering media with planar boundaries has been studied for nearly 100 years (Chandrasekar, 1960; Ambartsumian, 1943; de Hulst, 1980). This has been the model used for radiation processes in atmosphere. However, the boundaries involved in such problems are nonscattering in nature. Hence the fact that the scattering medium is confined to boundaries does not significantly affect the scattering processes. In several other situations where the boundaries are of scattering type, as in remote sensing of the earth (Elachi & van Zyl, 2006; Kuo & Moghaddam, 2007), seismology (Sato & Fehler, 1998), ground penetrating radar (Daniels, 2004; Urbini et al., 2001), optical devices (Amra, 1994; Elson, 1995), and medical tomography (Arridge & Hebden, 1997) the multiple scattering processes do get influenced by the boundaries. We study these processes in the context of a multilayer geometry in the following sections

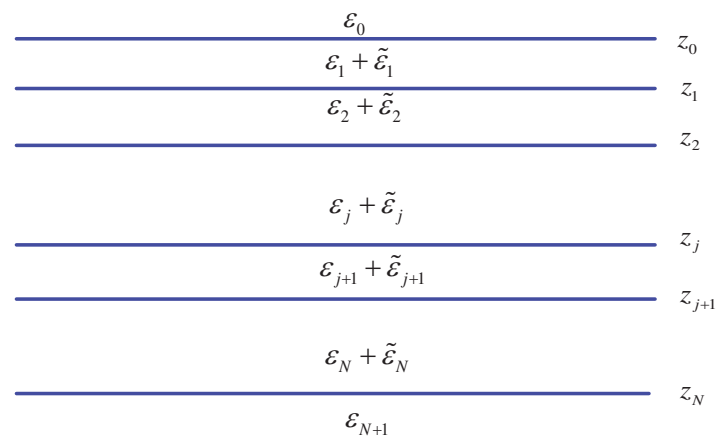


Fig. 2. Geometry of the problem with planar interfaces.

## 2.1 Description of the problem

The geometry of the problem is shown in Figure 2. We have an  $N$ -layer random media stack whose interfaces are parallel planes defined by  $z = z_0, z_1, z_2, \dots, z_N$ . The permittivity of the  $j$ -th layer is  $\epsilon_j + \tilde{\epsilon}_j(\mathbf{r})$ , where  $\epsilon_j$  is the deterministic part and  $\tilde{\epsilon}_j$  is the randomly fluctuating part. The permeability of each of the layers is that of free space. It is assumed that  $\tilde{\epsilon}_j$ s are zero-mean isotropic stationary random processes independent of each other. Let  $z_0 = 0$ , and let  $d_j$  be the thickness of the  $j$ -th layer. The media above and below the stack are homogeneous with parameters  $\epsilon_0, k_0$ , and  $\epsilon_{N+1}, k_{N+1}$ , respectively. This system is excited by a monochromatic electromagnetic plane wave and we are interested in the resulting multiply scattered fields.

## 2.2 Radiative transfer approach

Since our layer problem has translational invariance in azimuth the RT equation for the  $m$ -th layer takes the following form (Chandrasekar, 1960; Lenoble, 1993),

$$\cos \theta_m \frac{d}{dz} \mathbf{I}_m(z, \hat{s}) + \bar{\gamma}_m \mathbf{I}_m(z, \hat{s}) = \int_{\Omega_m} \bar{\mathbf{P}}_m(\hat{s}, \hat{s}') \mathbf{I}_m(z, \hat{s}') d\Omega', \quad (4)$$

where  $\Omega_m = \{\mathbf{r}'_{\perp}; z_m < z' < z_{m+1}\}$ . The subscript  $m$  denotes that the quantity corresponds to the  $m$ -th layer and  $\theta_m$  is the elevation angle of  $\hat{s}$  in the  $m$ -th layer. This set of RT equations is complemented by a set of boundary conditions (Karam & Fung, 1982; Caron et al., 2004) which are in turn based on energy conservation considerations. In other words, we impose the condition that the energy flux density at each interface is conserved. This leads to the following boundary condition on the  $m$ -th interface

$$\mathbf{I}_m^u(z_m, \hat{s}) = \mathcal{R}_{m+1,m}(\hat{s}) \mathbf{I}_m^d(z_m, \hat{s}) + \mathcal{T}_{m,m+1}(\hat{s}) \mathbf{I}_{m+1}^u(z_m, \hat{s}). \quad (5)$$

The boundary condition on the  $(m-1)$ -th interface is given as

$$\mathbf{I}_m^d(z_{m-1}, \hat{s}) = \mathcal{R}_{m-1,m}(\hat{s}) \mathbf{I}_m^u(z_{m-1}, \hat{s}) + \mathcal{T}_{m,m-1}(\hat{s}) \mathbf{I}_{m-1}^d(z_{m-1}, \hat{s}), \quad (6)$$

where  $\mathcal{R}_{mn}$  and  $\mathcal{T}_{mn}$  are the local reflection and transmission Müller matrices. To be more specific,  $\mathcal{R}_{mn}$  represents the reflection Müller matrix of waves incident from medium  $n$  on the interface that separates medium  $m$  and medium  $n$ . The superscripts  $u$  and  $d$  indicate whether the intensity corresponds to a wave travelling upwards or downwards. Suppose we have a

plane wave incident on this stack from above. Then the downward travelling intensity in Region 0 is

$$\mathbf{I}_0^d(z, \hat{s}) = \mathbf{B}_0 \delta(\cos \theta_0 - \cos \theta_i) \delta(\phi_0 - \phi_i), \quad (7)$$

where  $\mathbf{B}_0$  is the intensity of the incident plane wave and  $\{\theta_i, \phi_i\}$  describes its direction. Since there is no source or scatterer in Region  $N + 1$ ,

$$\mathbf{I}_{N+1}^u(z, \hat{s}) = 0.$$

Notice again that these boundary conditions represent conservation of intensity at the interfaces. In order to better understand this procedure we now relate this with the statistical wave approach to this problem.

### 2.3 Statistical wave approach

Following are the equations that govern the waves in the layer structure:

$$\nabla \times \nabla \times \mathbf{E}_j - k_j^2 \mathbf{E}_j = v_j \mathbf{E}_j \quad j = 1, \dots, N, \quad (8)$$

where  $v_j \equiv \omega^2 \mu \tilde{\epsilon}_j(\mathbf{r})$  represents the volumetric fluctuation in Region  $j$ . For the homogeneous regions above and below we have

$$\nabla \times \nabla \times \mathbf{E}_0 - k_0^2 \mathbf{E}_0 = 0, \quad (9a)$$

$$\nabla \times \nabla \times \mathbf{E}_{N+1} - k_{N+1}^2 \mathbf{E}_{N+1} = 0. \quad (9b)$$

The boundary conditions at the  $j$ -th interface are

$$\hat{\mathbf{z}} \times \mathbf{E}_j(\mathbf{r}_\perp, z_j) = \hat{\mathbf{z}} \times \mathbf{E}_{j+1}(\mathbf{r}_\perp, z_j), \quad (10a)$$

$$\hat{\mathbf{z}} \times \nabla \times \mathbf{E}_j(\mathbf{r}_\perp, z_j) = \hat{\mathbf{z}} \times \nabla \times \mathbf{E}_{j+1}(\mathbf{r}_\perp, z_j). \quad (10b)$$

This system is complemented by the radiation conditions well away from the stack. We assume that we know the solution to the unperturbed problem, and denote it as  $\mathbf{E}^0$  (Chew, 1995). The corresponding Green's functions, denoted as  $\bar{\mathbf{G}}_{ij}^0$ , are governed by the following set of equations:

$$\nabla \times \nabla \times \bar{\mathbf{G}}_{jk}^0(\mathbf{r}, \mathbf{r}') - k_j^2 \bar{\mathbf{G}}_{jk}^0(\mathbf{r}, \mathbf{r}') = \bar{\mathbf{I}} \delta_{jk} \delta(\mathbf{r} - \mathbf{r}'), \quad (11a)$$

$$\hat{\mathbf{z}} \times \bar{\mathbf{G}}_{jk}^0(\mathbf{r}_\perp, z_j; \mathbf{r}') = \hat{\mathbf{z}} \times \bar{\mathbf{G}}_{(j+1)k}^0(\mathbf{r}_\perp, z_j; \mathbf{r}'), \quad (11b)$$

$$\hat{\mathbf{z}} \times \nabla \times \bar{\mathbf{G}}_{jk}^0(\mathbf{r}_\perp, z_j; \mathbf{r}') = \hat{\mathbf{z}} \times \nabla \times \bar{\mathbf{G}}_{(j+1)k}^0(\mathbf{r}_\perp, z_j; \mathbf{r}'), \quad (11c)$$

$$\hat{\mathbf{z}} \times \bar{\mathbf{G}}_{jk}^0(\mathbf{r}_\perp, z_{j-1}; \mathbf{r}') = \hat{\mathbf{z}} \times \bar{\mathbf{G}}_{(j-1)k}^0(\mathbf{r}_\perp, z_{j-1}; \mathbf{r}'), \quad (11d)$$

$$\hat{\mathbf{z}} \times \nabla \times \bar{\mathbf{G}}_{jk}^0(\mathbf{r}_\perp, z_{j-1}; \mathbf{r}') = \hat{\mathbf{z}} \times \nabla \times \bar{\mathbf{G}}_{(j-1)k}^0(\mathbf{r}_\perp, z_{j-1}; \mathbf{r}'). \quad (11e)$$

$\bar{\mathbf{I}}$  represents the unit dyad. Using these Green's functions and the radiation conditions the electric field in Region  $j$  is represented as

$$\mathbf{E}_j(\mathbf{r}) = \mathbf{E}_j^0(\mathbf{r}) + \sum_{k=1}^N \int_{\Omega_k} \bar{\mathbf{G}}_{jk}^0(\mathbf{r}, \mathbf{r}') v_k(\mathbf{r}') \mathbf{E}_k(\mathbf{r}') d\mathbf{r}' \quad j = 0, 1, \dots, N + 1 \quad (12)$$

Note that  $v_0 = v_{N+1} = 0$ . We first average (10) w.r.t. fluctuation in permittivities to get

$$\langle \mathbf{E}_j(\mathbf{r}) \rangle_v = \mathbf{E}_j^0(\mathbf{r}) + \sum_{k=1}^N \sum_{l=1}^N \int_{\Omega_k} d\mathbf{r}' \int_{\Omega_l} d\mathbf{r}'' \bar{\mathbf{G}}_{jk}^0(\mathbf{r}, \mathbf{r}') \langle \bar{\mathbf{G}}_{kl}^0(\mathbf{r}', \mathbf{r}'') \rangle_v \langle v_k(\mathbf{r}') v_l(\mathbf{r}'') \rangle \langle \mathbf{E}_l(\mathbf{r}'') \rangle_v. \quad (13)$$



where  $\bar{\mathbf{G}}_{kl}$  is governed by the following system of equations

$$\begin{aligned}\nabla \times \nabla \times \bar{\mathbf{G}}_{kl}(\mathbf{r}, \mathbf{r}') - k_k^2 \bar{\mathbf{G}}_{kl}(\mathbf{r}, \mathbf{r}') &= \bar{\bar{\mathbf{I}}} \delta_{kl} \delta(\mathbf{r} - \mathbf{r}') + v_k \bar{\mathbf{G}}_{kl}(\mathbf{r}, \mathbf{r}'), \\ \hat{\mathbf{z}} \times \bar{\mathbf{G}}_{kl}(\mathbf{r}_\perp, z_k; \mathbf{r}') &= \hat{\mathbf{z}} \times \bar{\mathbf{G}}_{(k+1)l}(\mathbf{r}_\perp, z_k; \mathbf{r}'), \\ \hat{\mathbf{z}} \times \nabla \times \bar{\mathbf{G}}_{kl}(\mathbf{r}_\perp, z_k; \mathbf{r}') &= \hat{\mathbf{z}} \times \nabla \times \bar{\mathbf{G}}_{(k+1)l}(\mathbf{r}_\perp, z_k; \mathbf{r}').\end{aligned}$$

We also have a similar set of boundary conditions on the  $(k-1)$ -th interface. Here we have used a first-order approximation to the mass operator based on weak fluctuations. The fluctuations in permittivity in different regions are assumed to be uncorrelated, which means that

$$\langle v_k(\mathbf{r}') v_l(\mathbf{r}'') \rangle = \delta_{kl} C_k(\mathbf{r}' - \mathbf{r}''), \quad (14)$$

where  $C_k$  is the correlation function of the volumetric fluctuations in Region  $k$ . We have assumed that the fluctuations of the parameters of our problem are Gaussian and statistically homogeneous. Inserting (14) in (13) and employing  $\nabla \times \nabla \times \bar{\bar{\mathbf{I}}} - k_j^2$  on (13) we get

$$\nabla \times \nabla \times \langle \mathbf{E}_j(\mathbf{r}) \rangle - k_j^2 \langle \mathbf{E}_j(\mathbf{r}) \rangle_v = \int_{\Omega_j} d\mathbf{r}' \langle \bar{\mathbf{G}}_{jj}(\mathbf{r}, \mathbf{r}') \rangle C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle. \quad (15)$$

First note from (14) that  $(\nabla \times \nabla \times \bar{\bar{\mathbf{I}}} - k_j^2) \langle \mathbf{E}_j(\mathbf{r}) \rangle = 0$  for  $j = 0, N+1$ . This means that the coherent propagation constants in regions above and below the layer stack are unaffected by the fluctuations of the problem. However, they indeed get modified within the stack region.

On writing (15) as  $(\nabla \times \nabla \times \bar{\bar{\mathbf{I}}} - k_j^2 - \mathcal{L}) \langle \mathbf{E}_j \rangle = 0$ , where  $\mathcal{L}$  denotes the integral operator  $\int_{\Omega_j} d\mathbf{r}' \langle \bar{\mathbf{G}}_{jj}(\mathbf{r}, \mathbf{r}') \rangle C_j(\mathbf{r} - \mathbf{r}')$ , we infer that  $\chi_j \equiv \sqrt{k_j^2 + \mathcal{L}}$  represents the mean propagation constant in  $\bar{\Omega}_j$ . Observe that  $\chi_j$  depends explicitly on the volumetric fluctuations in Region  $j$  and implicitly on the fluctuations of the stack. This is in contrast to the RT approach where  $\bar{\gamma}_j$  depends exclusively on the volumetric fluctuations in Region  $j$ . Moreover,  $\chi_j$  depends on the polarization if the fluctuations of the problem are anisotropic. Further, even if the volumetric fluctuations are isotropic,  $\chi_j$  will be polarization-dependent because of surface reflections.

This is in contrast to the RT approach where  $\bar{\gamma}_j$  is polarization-dependent only when the volumetric fluctuations are anisotropic. Therefore the question is this: when do the effects of boundaries on the mean propagation constants become negligible? A first-order solution to the above dispersion relation shows that in situations where the thickness of the layer is larger than the corresponding mean free path, the influence of the boundaries on the mean propagation constants becomes negligible, as in the case of the RT system.

Since the problem is invariant under translations in azimuth, the mean wave functions for our problem have the following form:

$$\langle \mathbf{E}_j^p(\mathbf{r}) \rangle = \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) \left\{ A_j^p(\mathbf{k}_{\perp i}) \mathbf{p}_j^+ \exp[iq_j^p z] + B_j^p(\mathbf{k}_{\perp i}) \mathbf{p}_j^- \exp[-iq_j^p z] \right\} \quad j = 1, 2, \dots, N, \quad (16)$$

$$\langle \mathbf{E}_0^p(\mathbf{r}) \rangle = \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) \left\{ \mathbf{p}_0^- \exp[-ik_{0zi} z] + R^p(\mathbf{k}_{\perp i}) \mathbf{p}_0^+ \exp[ik_{0zi} z] \right\}, \quad (17)$$

and

$$\langle \mathbf{E}_{N+1}^p(\mathbf{r}) \rangle = \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) T^p(\mathbf{k}_{\perp i}) \mathbf{p}_{N+1}^- \exp[-ik_{(N+1)zi} z], \quad (18)$$

where the superscript  $p$  stands for the polarization, either horizontal or vertical.  $\mathbf{p}$  is the unit vector representing polarization.  $q_j$  is the  $z$ -component of  $\chi_j$ . The subscript  $i$  is used to indicate that the wave vector is in the incident direction.  $R$  and  $T$  denote, respectively, the mean reflection and transmission coefficients of the stack.  $A_j$  and  $B_j$  denote, respectively, the mean coefficients of up-going and down-going waves in the  $j$ -th layer. The boundary conditions associated with the above equations at the  $j$ -th interface are

$$\hat{\mathbf{z}} \times \langle \mathbf{E}_j(\mathbf{r}_\perp, z_j) \rangle = \hat{\mathbf{z}} \times \langle \mathbf{E}_{j+1}(\mathbf{r}_\perp, z_j) \rangle \quad j = 1, 2, \dots, N \quad (19a)$$

and

$$\hat{\mathbf{z}} \times \nabla \times \langle \mathbf{E}_j(\mathbf{r}_\perp, z_j) \rangle = \hat{\mathbf{z}} \times \nabla \times \langle \mathbf{E}_{j+1}(\mathbf{r}_\perp, z_j) \rangle \quad j = 1, 2, \dots, N. \quad (19b)$$

The above system may be solved to evaluate the mean coefficients that appear in (16)-(18).

We proceed now to the analysis of the second moments, by starting with (10). For convenience we write it in symbolic form as

$$\mathbf{E}_j = \mathbf{E}_j^o + \sum_{k=1}^N \bar{\mathbf{G}}_{jk}^o v_k \mathbf{E}_k. \quad (20)$$

We take the tensor product of this equation with its complex conjugate and average w.r.t. fluctuations in permittivity and obtain

$$\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \rangle = \langle \mathbf{E}_j \rangle \otimes \langle \mathbf{E}_j^* \rangle + \sum_{k=1}^N \sum_{k'=1}^N \sum_{l=1}^N \sum_{l'=1}^N \langle \bar{\mathbf{G}}_{jk} \rangle \otimes \langle \bar{\mathbf{G}}_{j'k'}^* \rangle \hat{\mathbf{K}}_{kk' ll'} \langle \mathbf{E}_l \otimes \mathbf{E}_{l'}^* \rangle, \quad (21)$$

where  $\hat{\mathbf{K}}$  is the intensity operator of the permittivity fluctuations. Employing the weak fluctuation approximation we approximate  $\hat{\mathbf{K}}$  by its leading term

$$\hat{\mathbf{K}}_{kk' ll'} \simeq \langle v_k \otimes v_k^* \rangle \delta_{kk' ll'} \bar{\mathbf{I}}. \quad (22)$$

The above is an equation for the second moment of the wave function  $\mathbf{E}$ , which can be decomposed into a coherent part  $\bar{\mathbf{E}}$  and a diffuse part  $\tilde{\mathbf{E}}$ . Therefore,

$$\langle \mathbf{E} \otimes \mathbf{E}^* \rangle = \langle \mathbf{E} \rangle \otimes \langle \mathbf{E}^* \rangle + \langle \tilde{\mathbf{E}} \otimes \tilde{\mathbf{E}}^* \rangle. \quad (23)$$

The coherent part is not of much interest, we know that it is specular for our problem. The diffuse or the incoherent part is of more interest. Therefore we write (21) in terms of the diffuse fields:

$$\langle \tilde{\mathbf{E}}_j \otimes \tilde{\mathbf{E}}_j^* \rangle = \sum_{k=1}^N \langle \bar{\mathbf{G}}_{jk} \rangle_v \otimes \langle \bar{\mathbf{G}}_{jk}^* \rangle_v \langle v_k \otimes v_k^* \rangle \langle \mathbf{E}_k \otimes \mathbf{E}_k^* \rangle, \quad (24)$$

Let us now write (24) in more detail as:

$$\begin{aligned} \langle \tilde{\mathbf{E}}_j(\mathbf{r}) \otimes \tilde{\mathbf{E}}_j^*(\mathbf{r}') \rangle &= \sum_{k=1}^N \int_{\Omega_k} d\mathbf{r}_1 \int_{\Omega_k} d\mathbf{r}'_1 \\ &\quad \langle \bar{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}_1) \rangle_v \otimes \langle \bar{\mathbf{G}}_{jk}^*(\mathbf{r}', \mathbf{r}'_1) \rangle_v |k_k|^4 C_k(\mathbf{r}_1 - \mathbf{r}'_1) \langle \mathbf{E}_k(\mathbf{r}_1) \otimes \mathbf{E}_k^*(\mathbf{r}'_1) \rangle. \end{aligned} \quad (25)$$

As it stands, this equation is not convenient for seeking a solution, either analytically or numerically. Besides, one important goal for us is to investigate the conditions needed for employing the radiative transfer approach. With this in mind, we introduce Wigner transforms. Note that (25) is an equation for the coherence function. On the other hand, the RT



equation, as we saw earlier, is an equation for the specific intensity, which is a 'phase-space' quantity. Wigner transforms serve as a bridge to link these two quantities (Yoshimori, 1998; Friberg, 1986; Marchand & Wolf, 1974; Pederson & Stamnes, 2000).

We introduce the Wigner transforms of waves and Green's functions as

$$\mathcal{E}_m\left(\frac{\mathbf{r}+\mathbf{r}'}{2}, \mathbf{k}\right) = \int \langle \mathbf{E}_m(\mathbf{r}) \otimes \mathbf{E}_m^*(\mathbf{r}') \rangle e^{-i\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}')} d(\mathbf{r}-\mathbf{r}'), \quad (26)$$

$$\mathcal{G}_{mn}\left(\frac{\mathbf{r}+\mathbf{r}'}{2}, \mathbf{k} \middle| \frac{\mathbf{r}_1+\mathbf{r}'_1}{2}, \mathbf{l}\right) = \int d(\mathbf{r}-\mathbf{r}') \int d(\mathbf{r}_1-\mathbf{r}'_1) e^{-i\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}')} e^{i\mathbf{l} \cdot (\mathbf{r}_1-\mathbf{r}'_1)} \langle \bar{\mathbf{G}}_{mn}(\mathbf{r}, \mathbf{r}_1) \rangle \otimes \langle \bar{\mathbf{G}}_{mn}^*(\mathbf{r}', \mathbf{r}'_1) \rangle. \quad (27)$$

In terms of these transforms, (25) becomes

$$\tilde{\mathcal{E}}_m(\mathbf{r}, \mathbf{k}) = \frac{1}{(2\pi)^6} \sum_{n=1}^N |k_n|^4 \int_{\Omega_n} d\mathbf{r}' \int d\alpha \int d\beta \mathcal{G}_{mn}(\mathbf{r}, \mathbf{k} | \mathbf{r}', \alpha) \Phi_n(\alpha - \beta) \mathcal{E}_n(\mathbf{r}', \beta), \quad (28)$$

where  $\Phi_n$  is the spectral density of the permittivity fluctuations in the  $n$ -th layer.

The fact that our problem has translational invariance in azimuth implies the following:

$$\mathcal{E}_m(\mathbf{r}, \mathbf{k}) = \mathcal{E}_m(z, \mathbf{k}), \quad (29a)$$

$$\mathcal{G}_{mn}(\mathbf{r}, \mathbf{k} | \mathbf{r}', \mathbf{l}) = \mathcal{G}_{mn}(z, \mathbf{k} | z', \mathbf{l}; \mathbf{r}_\perp - \mathbf{r}'_\perp). \quad (29b)$$

Using these relations in (28) we have

$$\tilde{\mathcal{E}}_m(z, \mathbf{k}) = \frac{1}{(2\pi)^6} \sum_{n=1}^N |k_n|^4 \int_{z_n}^{z_{n-1}} dz' \int d\alpha \int d\beta \mathcal{G}_{mn}(z, \mathbf{k} | z', \alpha; 0) \Phi_n(\alpha - \beta) \mathcal{E}_n(z', \beta), \quad (30)$$

where  $\mathcal{G}_{mn}(z, \mathbf{k} | z', \alpha; 0)$  is the Fourier transform of  $\mathcal{G}_{mn}(z, \mathbf{k} | z', \alpha; \mathbf{r}_\perp - \mathbf{r}'_\perp)$  w.r.t.  $\mathbf{r}_\perp - \mathbf{r}'_\perp$  evaluated at the origin of the spectral space. To proceed further we need to evaluate  $\mathcal{G}_{mn}$ . Furthermore, we need to relate this system to that of the RT, which involves the boundary conditions at the interfaces. In view of this we need to identify the coherence functions corresponding to up- and down-going wave functions. To facilitate this, we decompose  $\langle \bar{\mathbf{G}}_{mn} \rangle$  into its components,

$$\langle \bar{\mathbf{G}}_{mn} \rangle = \delta_{mn} \bar{\mathbf{G}}_m^o + \bar{\mathbf{G}}_{mn}^{uu} + \bar{\mathbf{G}}_{mn}^{ud} + \bar{\mathbf{G}}_{mn}^{du} + \bar{\mathbf{G}}_{mn}^{dd}, \quad (31)$$

where the first term is the singular part of the Green's function. The superscripts  $u$  and  $d$  indicate up- and down-going elements of the waves. The other components are due to reflections from boundaries. These are formally constructed using the concept of surface scattering operators as (Voronovich, 1999),

$$\langle G_{mn}^{ab}(\mathbf{r}, \mathbf{r}') \rangle^{\mu\nu} = \frac{1}{(2\pi)^4} \int \{ S_{mn}^{ab}(\mathbf{k}_\perp) \}^{\mu\nu} e^{i\mathbf{k}_\perp \cdot \mathbf{r} + iaq_m^\mu(\mathbf{k}_\perp)z} e^{-i\mathbf{k}_\perp \cdot \mathbf{r}' - ibq_n^\nu(\mathbf{k}_\perp)z'} d\mathbf{k}_\perp, \quad (32)$$

where  $\bar{S}_{mn}^{ab}$  is the surface scattering operator. The superscripts  $a$  and  $b$  on  $S$  are used to indicate whether the waves are up-going or down-going. In the exponents,  $a, b = 1$  if the waves are up-going. We let  $a, b = -1$  if the waves are down-going. The  $z$ -component of the mean propagation constant in the  $n$ -th layer is denoted as  $q_n$ .

We recall that  $\mathcal{G}_{mn}$  is the Wigner transform of  $\langle \bar{\mathbf{G}}_{mn} \rangle \otimes \langle \bar{\mathbf{G}}_{mn}^* \rangle$ . The superscripts  $\mu, \nu$  stand for polarization, either  $h$  or  $v$ . It is important to note that only in the quasi-uniform limit does the Wigner transform of the coherence function lead to the specific intensity of the RT equation. For our layer geometry, the Green's function is nonuniform. However, each of its components given in (31) is quasi-uniform. When we use (31) to perform the Wigner transform we ignore all cross terms. In other words, we make the approximation

$$\mathcal{G}_{mn} \simeq \delta_{mn} \mathcal{G}_m^o + \mathcal{G}_{mn}^{uu} + \mathcal{G}_{mn}^{ud} + \mathcal{G}_{mn}^{du} + \mathcal{G}_{mn}^{dd},$$

where  $\mathcal{G}_{mn}^{ab}$  is the Wigner transform of  $\langle \bar{\mathbf{G}}_{mn}^{ab} \rangle \otimes \langle \bar{\mathbf{G}}_{mn}^{ab*} \rangle$ . Most of the cross terms are nonuniform and may be neglected under the quasi-uniform field assumption. Two of the cross terms are quasi-uniform and their inclusion leads to phase matrices that are different from those of the RT system. It turns out that such additional coherence terms become negligible when the layer thickness is of the same order or greater than the mean free path of the corresponding layer. It is under these conditions that the approximate expression for  $\mathcal{G}_{mn}$  given above is good.

With the introduction of this representation for  $\mathcal{G}_{mn}$  in (28), we can trace the upward travelling and downward travelling waves to obtain the following equations for the coherence function:

$$\begin{aligned} \tilde{\mathcal{E}}_m^u(z, \mathbf{k}) &= \frac{1}{(2\pi)^6} |k_m|^4 \int_{z_m}^z dz' \int d\alpha \int d\beta \mathcal{G}_m^>(z, \mathbf{k}|z', \alpha; 0) \Phi_m(\alpha - \beta) \mathcal{E}_m(z', \beta) \\ &\quad + \frac{1}{(2\pi)^6} \sum_{n=1}^N |k_n|^4 \int_{z_n}^{z_{n-1}} dz' \int d\alpha \int d\beta \mathcal{G}_{mn}^{ua}(z, \mathbf{k}|z', \alpha; 0) \Phi_n(\alpha - \beta) \mathcal{E}_n^a(z', \beta) \end{aligned} \quad (33a)$$

$$\begin{aligned} \tilde{\mathcal{E}}_m^d(z, \mathbf{k}) &= \frac{1}{(2\pi)^6} |k_m|^4 \int_z^{z_{m-1}} dz' \int d\alpha \int d\beta \mathcal{G}_m^<(z, \mathbf{k}|z', \alpha; 0) \Phi_m(\alpha - \beta) \mathcal{E}_m(z', \beta) \\ &\quad + \frac{1}{(2\pi)^6} \sum_{n=1}^N |k_n|^4 \int_{z_n}^{z_{n-1}} dz' \int d\alpha \int d\beta \mathcal{G}_{mn}^{da}(z, \mathbf{k}|z', \alpha; 0) \Phi_n(\alpha - \beta) \mathcal{E}_n^a(z', \beta) \end{aligned} \quad (33b)$$

where  $\mathcal{G}_m^<$  and  $\mathcal{G}_m^>$  are the Wigner transforms of the tensor product of  $\bar{\mathbf{G}}_m^o$  when  $z < z'$  and  $z > z'$ , respectively. Note that summation over  $a = \{u, d\}$  is implied in the above equations. When we substitute the expressions for  $\mathcal{G}_{mn}$  in (33) we find that

$$\{\tilde{\mathcal{E}}_m^a(z, \mathbf{k})\}_{\mu\nu} = 2\pi\delta \left\{ k_z - \frac{1}{2}a[q_\mu^\mu(\mathbf{k}_\perp) + q_\nu^{\nu*}(\mathbf{k}_\perp)] \right\} e^{ia[q_\mu^\mu - q_\nu^{\nu*}]z} \{\tilde{\mathcal{E}}_m^a(z, \mathbf{k}_\perp)\}_{\mu\nu}. \quad (34)$$

On substituting this in (33) and differentiating w.r.t.  $z$  we obtain the following transport equations:

$$\begin{aligned} \left\{ \frac{d}{dz} - i[q_\mu(\mathbf{k}_\perp) - q_\nu^*(\mathbf{k}_\perp)] \right\} \tilde{\mathcal{E}}_{\mu\nu}^u(z, \mathbf{k}_\perp) &= \tilde{\mathcal{E}}_{\mu\nu}^u(z, \mathbf{k}_\perp) + \frac{|k_m|^4}{(2\pi)^2} \int d\alpha_\perp S_\mu^> S_\nu^{>*} \\ &\quad \Phi_m \left\{ \mathbf{k}_\perp - \alpha_\perp; \frac{1}{2}[q_\mu(\mathbf{k}_\perp) + q_\nu^*(\mathbf{k}_\perp)] - \frac{1}{2}a[q_{\mu'}(\alpha_\perp) + q_{\nu'}^*(\alpha_\perp)] \right\} (\boldsymbol{\mu} \cdot \boldsymbol{\mu}')(\boldsymbol{\nu} \cdot \boldsymbol{\nu}') \tilde{\mathcal{E}}_{\mu'\nu'}^a(z, \alpha_\perp), \end{aligned} \quad (35a)$$

$$\begin{aligned} \left\{ -\frac{d}{dz} - i[q_\mu(\mathbf{k}_\perp) - q_\nu^*(\mathbf{k}_\perp)] \right\} \tilde{\mathcal{E}}_{\mu\nu}^d(z, \mathbf{k}_\perp) &= \tilde{\mathcal{E}}_{\mu\nu}^d(z, \mathbf{k}_\perp) + \frac{|k_m|^4}{(2\pi)^2} \int d\alpha_\perp S_\mu^< S_\nu^{<*} \\ &\quad \Phi_m \left\{ \mathbf{k}_\perp - \alpha_\perp; -\frac{1}{2}[q_\mu(\mathbf{k}_\perp) + q_\nu^*(\mathbf{k}_\perp)] - \frac{1}{2}a[q_{\mu'}(\alpha_\perp) + q_{\nu'}^*(\alpha_\perp)] \right\} (\boldsymbol{\mu} \cdot \boldsymbol{\mu}')(\boldsymbol{\nu} \cdot \boldsymbol{\nu}') \tilde{\mathcal{E}}_{\mu'\nu'}^a(z, \alpha_\perp), \end{aligned} \quad (35b)$$

where  $\bar{\mathcal{E}}_{\mu\nu}^a$  represents scattering due to the coherent part of  $\mathcal{E}$ , whereas the integral term in (35) represents scattering due to the diffuse part of  $\mathcal{E}$ . We may also regard  $\bar{\mathcal{E}}_{\mu\nu}^a$  as the source to our transport equations and calculate it to obtain

$$\bar{\mathcal{E}}_{\mu\nu}^a = |k_m|^4 \Phi_m \left\{ \mathbf{k}_\perp - \mathbf{k}_{\perp i}; \frac{1}{2}a [q_\mu(\mathbf{k}_\perp) + q_\nu^*(\mathbf{k}_\perp)] - \frac{1}{2}b [q_\mu(\mathbf{k}_{\perp i}) + q_\nu^*(\mathbf{k}_{\perp i})] \right\} \times \\ \times \left( S_\mu^a S_\mu^{a*} \right) (\boldsymbol{\mu} \cdot \boldsymbol{\mu}_i) (\boldsymbol{\nu} \cdot \boldsymbol{\nu}_i) \left( (S_{m0}^b)_{\mu i} E_{\mu i} \right) \left( (S_{m0}^b)_{\nu i} E_{\nu i} \right)^*, \quad (36)$$

where summation over  $b$  is implied. Note that  $\bar{\mathcal{E}}^a$  in (35) includes both  $\bar{\mathcal{E}}^u$  and  $\bar{\mathcal{E}}^d$  (corresponding to up- and down-going waves). When the superscripts  $\{a, b\}$  correspond to  $u$ , the value of  $\{a, b\}$  in the argument of  $\Phi_m$  takes the value  $+1$ ; on the other hand, when the superscripts  $\{a, b\}$  correspond to  $d$  the value of  $a$  in the argument of  $\Phi_m$  takes the value  $-1$ . Since all quantities in (35) and (36) correspond to the same layer  $m$ , we have dropped the subscript  $m$  to avoid cumbersome notations. Summation over  $\mu'$  and  $\nu'$  is implicit in (35). Similarly, summation over  $\mu_i$  and  $\nu_i$  is implicit in (36).  $S_{m0}^b$  is the scattering amplitude of waves with direction  $b$  in  $m$ -th layer due to the wave incident in Region 0. To obtain appropriate boundary conditions we have to go back to the integral equation representations for  $\bar{\mathcal{E}}_{\mu\nu}^u$  and  $\bar{\mathcal{E}}_{\mu\nu}^d$ , observe their behaviour at the interfaces, and try to find a relation between them.

After some manipulations we arrived at the following boundary conditions. At the  $(m-1)$ -th interface we have

$$\bar{\mathcal{E}}_m^d(z_{m-1}, \mathbf{k}_\perp) = \ddot{\mathcal{R}}_{m-1,m}(\mathbf{k}_\perp) \bar{\mathcal{E}}_m^u(z_{m-1}, \mathbf{k}_\perp), \quad (37a)$$

with  $\ddot{\mathcal{R}} = \ddot{\mathbf{R}} \otimes \ddot{\mathbf{R}}^*$  where  $\ddot{\mathbf{R}}_{m-1,m}$  is the stack reflection matrix (not the local reflection matrix) for a wave incident from below on the  $(m-1)$ -th interface. Similarly,

$$\bar{\mathcal{E}}_m^u(z_{m-1}, \mathbf{k}_\perp) = \ddot{\mathcal{R}}_{m+1,m}(\mathbf{k}_\perp) \bar{\mathcal{E}}_m^d(z_{m-1}, \mathbf{k}_\perp), \quad (37b)$$

where  $\ddot{\mathcal{R}}_{m+1,m}$  is the tensor product of the stack reflection matrix for a wave incident from above on the  $m$ -th interface.

We were able to obtain the boundary conditions only after imposing certain approximations as given below. Consider the following identity:

$$\bar{\mathbf{S}}_{mm}^{du} = \bar{\mathbf{F}}_m \ddot{\mathbf{R}}_{m-1,m} \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \} \bar{\mathbf{F}}_m \quad (38)$$

where  $\bar{\mathbf{F}}_m = \text{diag} \{ e^{iq_n d_m}, e^{iq_v d_m} \}$ . Taking the tensor product of (38) with its complex conjugate we have

$$\bar{\mathbf{S}}_{mm}^{du} \otimes \bar{\mathbf{S}}_{mm}^{du*} = (\bar{\mathbf{F}}_m \otimes \bar{\mathbf{F}}_m^*) \left( \ddot{\mathbf{R}}_{m-1,m} \otimes \ddot{\mathbf{R}}_{m-1,m}^* \right) \left( \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \} \otimes \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \}^* \right) (\bar{\mathbf{F}}_m \otimes \bar{\mathbf{F}}_m^*). \quad (39)$$

A further approximation that we impose is given as follows:

$$\{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \} \otimes \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \}^* \simeq \bar{\mathbf{S}}_m^> \otimes \bar{\mathbf{S}}_m^{>*} + \bar{\mathbf{S}}_{mm}^{uu} \otimes \bar{\mathbf{S}}_{mm}^{uu*}. \quad (40)$$

This is similar to the approximation we used while computing the Wigner transforms of the Green's functions. We again need to use this approximation to arrive at our boundary conditions.

## 2.4 Transition to radiative transfer

Now we have to transition from this transport equation (40) to the phenomenological radiative transfer equation discussed earlier. To accomplish this we have to link the key quantities of waves and radiative transfer, viz., coherence function and specific intensity. The relation between them is obtained by computing the energy density using the two concepts. Thus we have

$$\frac{1}{2}\epsilon \left\{ \langle |E_v(\mathbf{r})|^2 \rangle + \langle |E_h(\mathbf{r})|^2 \rangle \right\} = \frac{1}{c} \int d\Omega_s I(\mathbf{r}, \hat{s}). \quad (41)$$

The Wigner transform provides us with the following relation:

$$\langle E_\mu(\mathbf{r}) E_\nu^*(\mathbf{r}) \rangle = \frac{1}{(2\pi)^2} \int d\mathbf{k}_\perp \mathcal{E}_{\mu\nu}(z, \mathbf{k}_\perp). \quad (42)$$

Defining  $\mathcal{I}_{\mu\nu}$  as

$$\mathcal{I}_{\mu\nu}(z, \hat{s}) = \frac{1}{2\eta} \frac{k'^2}{(2\pi)^2} \cos\theta \mathcal{E}_{\mu\nu}(z, \mathbf{k}_\perp), \quad (43)$$

where  $\eta$  is the intrinsic impedance of the medium, we have from (41) and (42)  $I = \mathcal{I}_{vv} + \mathcal{I}_{hh}$ . To facilitate comparison with the results of Ulaby et al. (Ulaby et al., 1986), and Lam and Ishimaru (Lam & Ishimaru, 1993) we use a modified version of the Stokes vector (Ishimaru, 1997). Instead of the standard form  $\{I, Q, U, V\}$  we use  $\{(I + Q)/2, (I - Q)/2, U, V\}$ . Thus, in terms of  $\mathcal{I}_{\mu\nu}$  defined in (43), our modified Stokes vector is  $\{\mathcal{I}_{vv}, \mathcal{I}_{hh}, \frac{1}{2}(\mathcal{I}_{vh} + \mathcal{I}_{hv}), -\frac{i}{2}(\mathcal{I}_{vh} - \mathcal{I}_{hv})\}$ . There is still one difference that needs to be ironed out before we transition to the RT equations. Notice that in our wave approach we obtained transport equations for  $\mathcal{E}$ , which is the fluctuating part of the coherence function. On the other hand, the phenomenological RT equations are traditionally written for total intensities. Therefore, we have to express our transport equations in terms of  $\mathcal{E}$ . Notice that  $\mathcal{E} = \bar{\mathcal{E}} + \tilde{\mathcal{E}}$ , where  $\bar{\mathcal{E}}$ , the average part of  $\mathcal{E}$ , satisfies:

$$\left\{ \frac{d}{dz} - ia(q_\mu - q_\nu^*) \right\} \bar{\mathcal{E}}_m^a(z, \mathbf{k}_\perp) = 0. \quad (44)$$

Using (44) in (35) we obtain

$$\begin{aligned} \left\{ \frac{d}{dz} - i[q_\mu(\mathbf{k}_\perp) - q_\nu^*(\mathbf{k}_\perp)] \right\} \mathcal{E}_{\mu\nu}^u(z, \mathbf{k}_\perp) &= \frac{|k_m|^4}{(2\pi)^2} \int d\alpha_\perp S_\mu^> S_\nu^{>*}(\boldsymbol{\mu} \cdot \boldsymbol{\mu}')(\boldsymbol{\nu} \cdot \boldsymbol{\nu}') \times \\ &\times \Phi_m \left\{ \mathbf{k}_\perp - \alpha_\perp; \frac{1}{2}[q_\mu(\mathbf{k}_\perp) - q_\nu^*(\mathbf{k}_\perp)] - \frac{1}{2}a[q_{\mu'}(\alpha_\perp) - q_{\nu'}^*(\alpha_\perp)] \right\} \mathcal{E}_{\mu'\nu'}^a(z, \alpha_\perp), \end{aligned} \quad (45a)$$

$$\begin{aligned} \left\{ -\frac{d}{dz} - i[q_\mu(\mathbf{k}_\perp) - q_\nu^*(\mathbf{k}_\perp)] \right\} \mathcal{E}_{\mu\nu}^d(z, \mathbf{k}_\perp) &= \frac{|k_m|^4}{(2\pi)^2} \int d\alpha_\perp S_\mu^< S_\nu^{<*}(\boldsymbol{\mu} \cdot \boldsymbol{\mu}')(\boldsymbol{\nu} \cdot \boldsymbol{\nu}') \times \\ &\times \Phi_m \left\{ \mathbf{k}_\perp - \alpha_\perp; -\frac{1}{2}[q_\mu(\mathbf{k}_\perp) - q_\nu^*(\mathbf{k}_\perp)] - \frac{1}{2}a[q_{\mu'}(\alpha_\perp) - q_{\nu'}^*(\alpha_\perp)] \right\} \mathcal{E}_{\mu'\nu'}^a(z, \alpha_\perp). \end{aligned} \quad (45b)$$

Notice that this equation is expressed entirely in total intensity. Now we can transition to the phenomenological RT equations. Using the relation between  $\mathcal{E}$  and  $I$ , we change the integration variable to solid angle and arrive at

$$\left\{ \cos\theta \frac{d}{dz} + \gamma_{ij} \right\} I_j^u(z, \hat{s}) = \int P_{ij}^{ua}(\Omega, \Omega') I_j^a(z, \hat{s}') d\Omega', \quad (46a)$$

$$\left\{ -\cos\theta \frac{d}{dz} + \gamma_{ij} \right\} I_j^d(z, \hat{s}) = \int P_{ij}^{da}(\Omega, \Omega') I_j^a(z, \hat{s}') d\Omega', \quad (46b)$$

where  $\bar{\gamma}$  is the extinction matrix and  $\bar{\mathbf{P}}$  is the phase matrix. Implicit summation over superscripts  $a$  and subscript  $j$  is assumed in (46). Although the structure of this equation is identical to that of the RT (Equation (2)), the elements of the phase matrix and the extinction matrices are not the same primarily because of coherence induced by the boundaries. As mentioned earlier, we assume that the layer thickness is greater than the mean free path of the corresponding medium. If, in addition, we assume the quasi-homogeneous field approximation we obtain the following expressions for the extinction and phase matrices:

$$\bar{\gamma} = \cos\theta \text{diag} \{ 2q_v'', 2q_h'', q_v'' + q_h'', q_v'' + q_h'' \} \quad (47a)$$

$$P_{\mu\nu}^{ab} = \frac{1}{(2\pi)^2} \frac{1}{4} |k_m|^4 \Phi_m \{ \mathbf{k}_\perp - \mathbf{k}'_\perp; k'_m [a \cos\theta - b \cos\theta'] \} \mathcal{P}_{\mu\nu}^{ab}. \quad (47b)$$

The double primes are used in (47) to denote imaginary parts. For  $\mu = \{v, h\}$ ,

$$\begin{aligned} \mathcal{P}_{\mu v}^{ab} &= (\boldsymbol{\mu}^a \cdot \mathbf{v}'^b)^2 & \mathcal{P}_{\mu h}^{ab} &= (\boldsymbol{\mu}^a \cdot \mathbf{h}'^b)^2 \\ \mathcal{P}_{\mu U}^{ab} &= (\boldsymbol{\mu}^a \cdot \mathbf{v}'^b) (\boldsymbol{\mu}^a \cdot \mathbf{h}'^b) & \mathcal{P}_{\mu V}^{ab} &= 0 \end{aligned} \quad (48)$$

Similarly,

$$\begin{aligned} \mathcal{P}_{Uv}^{ab} &= 2 (\mathbf{v}^a \cdot \mathbf{v}'^b) (\mathbf{h}^a \cdot \mathbf{v}'^b) & \mathcal{P}_{Uh}^{ab} &= 2 (\mathbf{v}^a \cdot \mathbf{h}'^b) (\mathbf{h}^a \cdot \mathbf{h}'^b) \\ \mathcal{P}_{UU}^{ab} &= (\mathbf{v}^a \cdot \mathbf{v}'^b) (\mathbf{h}^a \cdot \mathbf{h}'^b) + (\mathbf{v}^a \cdot \mathbf{h}'^b) (\mathbf{h}^a \cdot \mathbf{v}'^b) & \mathcal{P}_{UV}^{ab} &= 0 \end{aligned} \quad (49)$$

$$\mathcal{P}_{Vv}^{ab} = \mathcal{P}_{Vh}^{ab} = \mathcal{P}_{VU}^{ab} = 0$$

$$\mathcal{P}_{VV}^{ab} = (\mathbf{v}^a \cdot \mathbf{v}'^b) (\mathbf{h}^a \cdot \mathbf{h}'^b) - (\mathbf{v}^a \cdot \mathbf{h}'^b) (\mathbf{h}^a \cdot \mathbf{v}'^b) \quad (50)$$

Noting the implied summation over  $a$  in (46) we see that they are identical to the RT equations given in Section 2. Now we have explicit expressions for the extinction matrix and phase matrix in terms of the statistical parameters of the problem, thanks to our wave approach.

We next turn our attention to the boundary conditions (BC). In our statistical wave approach we obtained BCs in terms of the 'stack' reflection matrix  $\bar{\bar{\mathbf{R}}}$ , whereas in the RT approach the BCs are given in terms of the local interface reflection matrices. We can readily reconcile this apparent difference. Note that the BC in the wave approach forms a closed system whereas in the RT approach it is 'open' (linked to adjacent layer intensities). Let us take a look at the BC at the  $(m-1)$ -th interface.  $\bar{\bar{\mathbf{R}}}_{m-2,m}$  can be expressed in terms of  $\bar{\mathbf{R}}_{m-2,m-1}$  as follows,

$$\bar{\bar{\mathbf{R}}}_{m-1,m} = \bar{\mathbf{R}}_{m-1,m} + \bar{\mathbf{T}}_{m,m-1} \left\{ I - \bar{\bar{\mathbf{R}}}_{m-2,m-1} \bar{\mathbf{F}}_{m-1} \bar{\mathbf{R}}_{m,m-1} \right\}^{-1} \bar{\bar{\mathbf{R}}}_{m-2,m-1} \bar{\mathbf{F}}_{m-1} \bar{\mathbf{T}}_{m-1,m}. \quad (51)$$

This is the relation between the stack reflection coefficients of adjacent interfaces. The  $\bar{\mathbf{R}}$  and  $\bar{\mathbf{T}}$  are local (single interface) reflection and transmission matrices at the  $(m-1)$ -th interface. On operating  $\mathbf{E}_m^u$  with (51) we get

$$\mathbf{E}_m^d = \bar{\mathbf{R}}_{m-1,m} \mathbf{E}_m^u + \bar{\mathbf{T}}_{m,m-1} \mathbf{E}_{m-1}^d. \quad (52)$$

Notice that this boundary condition now involves only local interface Fresnel coefficients.

Similarly we write  $\ddot{\mathbf{R}}_{m+1,m}$  in terms of  $\ddot{\mathbf{R}}_{m+2,m+1}$  and hence obtain the BC at the  $m$ -th interface as

$$\mathbf{E}_m^u = \bar{\mathbf{R}}_{m+1,m} \mathbf{E}_m^d + \bar{\mathbf{T}}_{m,m+1} \mathbf{E}_{m+1}^u. \quad (53)$$

Next we take the tensor product of (52) with its complex conjugate. Employing the Wigner transform operator to that product, we obtain the boundary condition at the  $(m-1)$ -th interface, which is similar to that of the RT system. However, the reflection and transmission matrices used in the RT system correspond to the unperturbed medium, as opposed to the average medium as in the case of the statistical wave approach. Similarly we can obtain the transport-theoretic boundary conditions at the  $m$ -th interface using (53).

## 2.5 Remarks

Now that we have made the transition from statistical wave theory to radiative transfer theory it is instructive to itemize the assumptions implicitly involved in the RT approach.

1. Quasi-stationary field approximation.
2. Weak fluctuations.
3. Statistical homogeneity of fluctuations.

These are the three well-known conditions necessary for the unbounded random media problem. However, if the medium is bounded we need to impose additional conditions. We found that the extinction coefficients calculated in the wave approach and the RT approach are different and only after applying further approximations can they be made to agree with each other. The following two additional conditions are required for our bounded random media problem:

4. Layer thickness must be of the same order or greater than the corresponding mean free path.
5. All fluctuations of the problem are statistically independent.

In the next section, we turn our attention to the problem where the interfaces are randomly rough.

## 3. Layered random media with rough interfaces

The model of layered random media with rough interfaces is often encountered in many applications in various disciplines. A simple approach is to incoherently add the contributions of volumetric and surface fluctuations (Zuniga et al., 1979; Lee & Kong, 1985). However, this is valid only when we are in the single-scattering regime (Elson, 1997; Mudaliar, 1994). There are some other hybrid approaches (Papa & Tamasanis, 1991; Chauhan et al., 1991) which take into consideration some multiple scattering effects. Brown (Brown, 1988) outlines an iterative procedure which properly includes all multiple scattering interactions. However, it does not appear feasible to carry out the calculation beyond one or two iterations. Among the other methods currently used, perhaps the most widely used approach is the radiative transfer (RT) approach (Ulaby et al., 1986; Lam & Ishimaru, 1993; Karam & Fung, 1982; Shin & Kong, 1989; Caron et al., 2004; Ulaby et al., 1990; Liang et al., 2005; Fung & Chen, 1981). Here, one formulates the scattering and propagation in each layer by using the radiative transfer equation, which involves only the parameters of the medium of that layer. The boundary conditions are derived separately and independently using some asymptotic procedure developed in rough surface scattering theory (Beckmann & Spizzichino, 1987; Bass & Fuks, 1979; Voronovich, 1999). The RT equations, along with the boundary conditions, comprise the



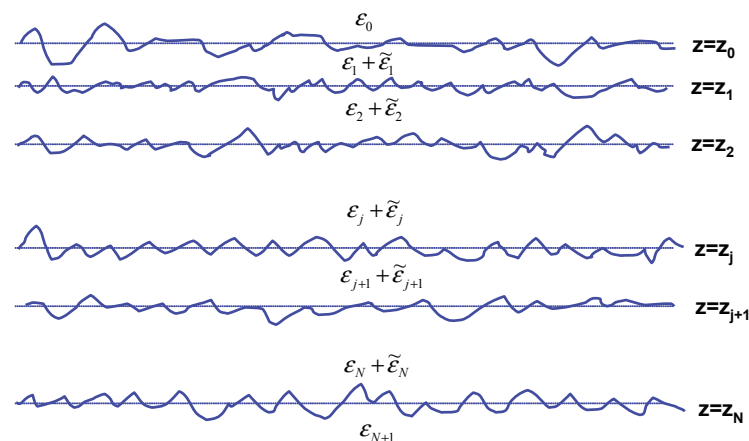


Fig. 3. Geometry of the problem with rough interfaces.

system that describes the problem. In order to better understand the conditions under which this procedure is good we study the following multi-layer problem.

### 3.1 Description of the problem

The geometry of the problem is shown in Figure 3. All the parameters of this problem are the same as in Section 2 except the interfaces are randomly rough now. Thus we have an  $N$ -layer random media stack with rough interfaces. The randomly rough interfaces are defined as  $z = \zeta_j \equiv z_j + \tilde{\zeta}_j(\mathbf{r}_\perp)$ . It is assumed that  $\tilde{\epsilon}_j$  and  $\tilde{\zeta}_j$  are zero-mean isotropic stationary random processes independent of each other. Thus, on the average the interfaces are parallel planes defined as  $z = z_0, z_1, z_2, \dots, z_N$ . Let  $z_0 = 0$ , and let  $d_j$  be the thickness of the  $j$ -th layer. As before, this system is excited by a monochromatic electromagnetic plane wave and we are interested in formulating the resulting multiple scattering process.

### 3.2 Radiative transfer approach

The radiative transfer equations for this problem are the same as in the planar interface case (Equation (4)). However, the boundary conditions are different. On the  $m$ -th interface it is

$$\mathbf{I}_m^u(z_m, \hat{s}) = \int \langle \mathcal{R}_{m+1,m}(\hat{s}, \hat{s}') \rangle \mathbf{I}_m^d(z_m, \hat{s}') d\Omega' + \int \langle \mathcal{T}_{m,m+1}(\hat{s}, \hat{s}') \rangle \mathbf{I}_{m+1}^u(z_m, \hat{s}') d\Omega'. \quad (54)$$

The boundary condition on the  $(m-1)$ -th interface is given as

$$\mathbf{I}_m^d(z_{m-1}, \hat{s}) = \int \langle \mathcal{R}_{m-1,m}(\hat{s}, \hat{s}') \rangle \mathbf{I}_m^u(z_{m-1}, \hat{s}') d\Omega' + \int \langle \mathcal{T}_{m,m-1}(\hat{s}, \hat{s}') \rangle \mathbf{I}_{m-1}^d(z_{m-1}, \hat{s}') d\Omega', \quad (55)$$

where  $\mathcal{R}_{mn}$  and  $\mathcal{T}_{mn}$  are the local reflection and transmission Müller matrices. To be more specific,  $\mathcal{R}_{mn}$  represents the reflection Müller matrix of waves incident from medium  $n$  on the interface that separates medium  $m$  and medium  $n$ . The superscripts  $u$  and  $d$  indicate whether the intensity corresponds to a wave travelling upwards or downwards. The integrations in these expressions are over a solid angle (hemisphere) corresponding to  $\hat{s}'$ . For the time-harmonic plane wave incident on this stack from above the downward going intensity in Region 0 and the upward going intensity in Region  $N+1$  are given as

$$\mathbf{I}_0^d(z, \hat{s}) = \mathbf{B}_0 \delta(\cos \theta_0 - \cos \theta_i) \delta(\phi_0 - \phi_i), \quad (56)$$

$$\mathbf{I}_{N+1}^u(z, \hat{s}) = 0.$$

This is the system that is widely used for this kind of problem Ulaby et al. (1986). The validity conditions for this approach have never been clearly stated. Let us now see what we can learn from adopting a statistical wave approach to this problem.

### 3.3 Statistical wave approach

The governing equations for the electric fields are the same as in Section 2. The primary difference is in the boundary conditions, which on the  $j$ -th interface are

$$\hat{\mathbf{n}}_j \times \mathbf{E}_j(\mathbf{r}_\perp, \zeta_j) = \hat{\mathbf{n}}_j \times \mathbf{E}_{j+1}(\mathbf{r}_\perp, \zeta_j), \quad (57a)$$

$$\hat{\mathbf{n}}_j \times \nabla \times \mathbf{E}_j(\mathbf{r}_\perp, \zeta_j) = \hat{\mathbf{n}}_j \times \nabla \times \mathbf{E}_{j+1}(\mathbf{r}_\perp, \zeta_j). \quad (57b)$$

where  $\hat{\mathbf{n}}_j$  is the unit vector normal to the  $j$ -th interface with normal pointing into the medium  $j$ . This system is complemented by the radiation conditions well away from the stack. We assume that we know the solution to the problem without volumetric fluctuations, and denote it as  $\check{\mathbf{E}}$ . The corresponding Green's functions, denoted as  $\check{\mathbf{G}}_{ij}$ , are governed by the following set of equations:

$$\nabla \times \nabla \times \check{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}') - k_j^2 \check{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}') = \bar{\mathbf{I}} \delta_{jk} \delta(\mathbf{r} - \mathbf{r}'), \quad (58a)$$

$$\hat{\mathbf{n}}_j \times \check{\mathbf{G}}_{jk}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}') = \hat{\mathbf{n}}_j \times \check{\mathbf{G}}_{(j+1)k}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}'), \quad (58b)$$

$$\hat{\mathbf{n}}_j \times \nabla \times \check{\mathbf{G}}_{jk}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}') = \hat{\mathbf{n}}_j \times \nabla \times \check{\mathbf{G}}_{(j+1)k}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}'). \quad (58c)$$

Another pair of equations similar to (58b) and (58c) corresponding to the  $(j-1)$ -th interface must be added to this list. Using these Green's functions and the radiation conditions the wave functions can be represented as

$$\mathbf{E}_j(\mathbf{r}) = \check{\mathbf{E}}_j(\mathbf{r}) + \sum_{k=1}^N \int_{\Omega_k} d\mathbf{r}' \check{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}') v_k(\mathbf{r}') \mathbf{E}_k(\mathbf{r}') \quad j = 0, 1, \dots, N+1 \quad (59)$$

where  $\Omega_k = \{\mathbf{r}'_\perp; \zeta_k < z' < \zeta_{k+1}\}$ . We first average (59) w.r.t. volumetric fluctuations to get

$$\langle \mathbf{E}_j(\mathbf{r}) \rangle_v = \check{\mathbf{E}}_j(\mathbf{r}) + \sum_{k=1}^N \sum_{l=1}^N \int_{\Omega_k} d\mathbf{r}' \int_{\Omega_l} d\mathbf{r}'' \check{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}') \langle \bar{\mathbf{G}}_{kl}(\mathbf{r}', \mathbf{r}'') \rangle_v \langle v_k(\mathbf{r}') v_l(\mathbf{r}'') \rangle \langle \mathbf{E}_l(\mathbf{r}'') \rangle_v, \quad (60)$$

where  $\bar{\mathbf{G}}_{kl}$  is governed by the following system of equations:

$$\nabla \times \nabla \times \bar{\mathbf{G}}_{kl}(\mathbf{r}, \mathbf{r}') - k_k^2 \bar{\mathbf{G}}_{kl}(\mathbf{r}, \mathbf{r}') = \bar{\mathbf{I}} \delta_{kl} \delta(\mathbf{r} - \mathbf{r}') + v_k \bar{\mathbf{G}}_{kl}(\mathbf{r}, \mathbf{r}'),$$

$$\hat{\mathbf{n}}_k \times \bar{\mathbf{G}}_{kl}(\mathbf{r}_\perp, \zeta_k; \mathbf{r}') = \hat{\mathbf{n}}_k \times \bar{\mathbf{G}}_{(k+1)l}(\mathbf{r}_\perp, \zeta_k; \mathbf{r}'),$$

$$\hat{\mathbf{n}}_k \times \nabla \times \bar{\mathbf{G}}_{kl}(\mathbf{r}_\perp, \zeta_k; \mathbf{r}') = \hat{\mathbf{n}}_k \times \nabla \times \bar{\mathbf{G}}_{(k+1)l}(\mathbf{r}_\perp, \zeta_k; \mathbf{r}').$$

Here,  $\hat{\mathbf{n}}_k$  is the unit vector normal to the  $k$ -th interface. We also have a similar set of boundary conditions on the  $(k-1)$ -th interface. The subscript  $v$  is used to denote averaging with respect to volumetric fluctuations. Here, we have used a first-order approximation to the mass operator based on weak permittivity fluctuations. We have assumed that the fluctuations of the parameters of our problem are Gaussian and statistically homogeneous. Imposing the

condition that the volumetric fluctuations in different regions are uncorrelated, by using (14) in (60) and employing  $\nabla \times \nabla \times \bar{\mathbf{I}} - k_j^2$  on (60) we get

$$\nabla \times \nabla \times \langle \mathbf{E}_j(\mathbf{r}) \rangle_v - k_j^2 \langle \mathbf{E}_j(\mathbf{r}) \rangle_v = \int_{\Omega_j} \langle \bar{\mathbf{G}}_{jj}(\mathbf{r}, \mathbf{r}') \rangle_v C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle_v d\mathbf{r}'. \quad (61)$$

Next we average (61) over the surface fluctuations,

$$\nabla \times \nabla \times \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} - k_j^2 \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} = \int_{\bar{\Omega}_j} \left\langle \langle \bar{\mathbf{G}}_{jj}(\mathbf{r}, \mathbf{r}') \rangle_v C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle_v \right\rangle_s d\mathbf{r}',$$

where the subscript  $s$  denotes averaging over surface fluctuations and  $\bar{\Omega}_j = \{\mathbf{r}'_{\perp}; \hat{\mathbf{z}}_j < z' < \hat{\mathbf{z}}_{j-1}\}$ . We approximate  $\left\langle \langle \bar{\mathbf{G}}_{jj}(\mathbf{r}, \mathbf{r}') \rangle_v C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle_v \right\rangle_s$  as  $\left\langle \bar{\mathbf{G}}_{jj}(\mathbf{r}, \mathbf{r}') \right\rangle_{vs} C_j(\mathbf{r} - \mathbf{r}') \left\langle \mathbf{E}_j(\mathbf{r}') \right\rangle_{vs}$  and obtain

$$\nabla \times \nabla \times \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} - k_j^2 \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} = \int_{\bar{\Omega}_j} \langle \bar{\mathbf{G}}_{jj}(\mathbf{r}, \mathbf{r}') \rangle_{vs} C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle_{vs} d\mathbf{r}'. \quad (62)$$

We call this the weak surface correlation approximation. We will later see that this is one additional approximation necessary to arrive at the RT system. As before, the result that  $(\nabla \times \nabla \times \bar{\mathbf{I}} - k_j^2) \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} = 0$  for  $j = 0, N + 1$  implies that the coherent propagation constants in regions above and below the layer stack are unaffected by the fluctuations of the problem. However, they indeed get modified within the stack region. On writing (62) as  $(\nabla \times \nabla \times \bar{\mathbf{I}} - k_j^2 - \mathcal{L}) \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} = 0$ , where  $\mathcal{L}$  denotes the integral operator  $\int_{\bar{\Omega}_j} d\mathbf{r}' \left\langle \bar{\mathbf{G}}_{jj}(\mathbf{r}, \mathbf{r}') \right\rangle_{vs} C_j(\mathbf{r} - \mathbf{r}')$ , we infer that  $\chi_j \equiv \sqrt{k_j^2 + \mathcal{L}}$  represents the mean propagation constant in  $\bar{\Omega}_j$ . Observe that  $\chi_j$  depends explicitly on the volumetric fluctuations in Region  $j$  and implicitly on the fluctuations of the stack, both volumetric and surface. This is in contrast to the RT approach where  $\bar{\gamma}_j$  depends exclusively on the volumetric fluctuations in Region  $j$ . Moreover,  $\chi_j$  depends on the polarization if the fluctuations of the problem are anisotropic. Further, even if the volumetric fluctuations are isotropic  $\chi_j$  will be polarization-dependent because of surface reflections. This is in contrast to the RT approach where  $\bar{\gamma}_j$  is polarization-dependent only when the volumetric fluctuations are anisotropic. A first-order solution to the above dispersion relation shows that in situations where the thickness of the layer is larger than the corresponding mean free path the influence of the boundaries on the mean propagation constants become negligible.

Since the problem is invariant under translations in azimuth the mean wave functions for our problem have the following form:

$$\langle \mathbf{E}_j^p(\mathbf{r}) \rangle_{vs} = \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) \left\{ A_j^p(\mathbf{k}_{\perp i}) \mathbf{p}_j^+ \exp[iq_j^p z] + B_j^p(\mathbf{k}_{\perp i}) \mathbf{p}_j^- \exp[-iq_j^p z] \right\} \quad j = 1, 2, \dots, N, \quad (63)$$

$$\langle \mathbf{E}_0^p(\mathbf{r}) \rangle_{vs} = \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) \left\{ \mathbf{p}_0^- \exp[-ik_{0zi} z] + R^p(\mathbf{k}_{\perp i}) \mathbf{p}_0^+ \exp[ik_{0zi} z] \right\}, \quad (64)$$

and

$$\langle \mathbf{E}_{N+1}^p(\mathbf{r}) \rangle_{vs} = \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) T^p(\mathbf{k}_{\perp i}) \mathbf{p}_{N+1}^- \exp[-ik_{(N+1)zi} z], \quad (65)$$

where the superscript  $p$  stands for the polarization, either horizontal or vertical.  $\mathbf{p}$  is the unit vector representing polarization.  $R$  and  $T$  denote, respectively, the mean reflection and transmission coefficients of the stack.  $A_j$  and  $B_j$  denote, respectively, the mean coefficients of up-going and down-going waves in the  $j$ -th layer. Based on this we can formulate the waves averaged w.r.t. volumetric fluctuations as

$$\langle \mathbf{E}_j(\mathbf{r}) \rangle_v^p = \frac{1}{4\pi^2} \int d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp \cdot \mathbf{r}) \left\{ A_j^{pq}(\mathbf{k}_\perp, \mathbf{k}_{\perp i}) \mathbf{q}_j^+ \exp[iq_j z] + B_j^{pq}(\mathbf{k}_\perp, \mathbf{k}_{\perp i}) \mathbf{q}_j^- \exp[-iq_j z] \right\} \quad j = 1, 2, \dots, N, \quad (66)$$

$$\langle \mathbf{E}_0(\mathbf{r}) \rangle_v^p = \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) \exp[-ik_{0zi}z] \mathbf{p}_0^- + \frac{1}{4\pi^2} \int d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp \cdot \mathbf{r}) R^{pq}(\mathbf{k}_\perp, \mathbf{k}_{\perp i}) \mathbf{q}_0^+ \exp[ik_{0z}z], \quad (67)$$

and

$$\langle \mathbf{E}_{N+1}(\mathbf{r}) \rangle_v^p = \frac{1}{4\pi^2} \int d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp \cdot \mathbf{r}) T^{pq}(\mathbf{k}_\perp, \mathbf{k}_{\perp i}) \mathbf{q}_{N+1}^- \exp[-ik_{(N+1)z}z], \quad (68)$$

where  $A_j$ ,  $B_j$ ,  $R$ , and  $T$  are now integral operators representing scattering from rough interfaces. The boundary conditions associated with the above equations at the  $j$ -th interface are

$$\hat{\mathbf{n}}_j \times \langle \mathbf{E}_j(\mathbf{r}_\perp, \zeta_j) \rangle_v = \hat{\mathbf{n}}_j \times \langle \mathbf{E}_{j+1}(\mathbf{r}_\perp, \zeta_j) \rangle_v \quad j = 1, 2, \dots, N \quad (69a)$$

and

$$\hat{\mathbf{n}}_j \times \nabla \times \langle \mathbf{E}_j(\mathbf{r}_\perp, \zeta_j) \rangle_v = \hat{\mathbf{n}}_j \times \nabla \times \langle \mathbf{E}_{j+1}(\mathbf{r}_\perp, \zeta_j) \rangle_v \quad j = 1, 2, \dots, N. \quad (69b)$$

The above system may be solved either numerically or by any of the asymptotic methods<sup>1</sup> available in rough surface scattering theory (Beckmann & Spizzichino, 1987; Bass & Fuks, 1979; Voronovich, 1999) to evaluate the mean coefficients that appear in (63)-(65).

We proceed now to the analysis of the second moments, by starting with (59) represented in symbolic form as

$$\mathbf{E}_j = \check{\mathbf{E}}_j + \sum_{k=1}^N \check{\mathbf{G}}_{jk} v_k \mathbf{E}_k. \quad (70)$$

We take the tensor product of this equation with its complex conjugate and average w.r.t. volumetric fluctuations and obtain

$$\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \rangle_v = \langle \mathbf{E}_j \rangle_v \otimes \langle \mathbf{E}_j^* \rangle_v + \sum_{k=1}^N \sum_{k'=1}^N \sum_{l=1}^N \sum_{l'=1}^N \langle \bar{\mathbf{G}}_{jk} \rangle_v \otimes \langle \bar{\mathbf{G}}_{j'k'}^* \rangle_v \hat{\mathbf{K}}_{kk' ll'} \langle \mathbf{E}_l \otimes \mathbf{E}_{l'}^* \rangle_v, \quad (71)$$

where  $\hat{\mathbf{K}}$  is the intensity operator of the volumetric fluctuations. Employing the weak fluctuation approximation we approximate  $\hat{\mathbf{K}}$  by its leading term

$$\hat{\mathbf{K}}_{kk' ll'} \simeq \langle v_k \otimes v_k^* \rangle \delta_{kk' ll'} \bar{\mathbf{I}}. \quad (72)$$

Next, we average (71) w.r.t. the surface fluctuations and employ the weak surface correlation approximation, as before, to get

$$\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \rangle_{vs} = \langle \langle \mathbf{E}_j \rangle_v \otimes \langle \mathbf{E}_j^* \rangle_v \rangle_s + \sum_{k=1}^N \left\langle \langle \bar{\mathbf{G}}_{jk} \rangle_v \otimes \langle \bar{\mathbf{G}}_{jk}^* \rangle_v \right\rangle_s \langle v_k \otimes v_k^* \rangle \langle \mathbf{E}_k \otimes \mathbf{E}_k^* \rangle_{vs}. \quad (73)$$

<sup>1</sup>It is necessary, however, to meet the weak surface correlation approximation employed earlier

The above is an equation for the second moment of the wave function  $\mathbf{E}$ , which can be decomposed into a coherent part  $\bar{\mathbf{E}}$  and a diffuse part  $\tilde{\mathbf{E}}$ . Therefore,

$$\langle \mathbf{E} \otimes \mathbf{E}^* \rangle = \langle \mathbf{E} \rangle \otimes \langle \mathbf{E}^* \rangle + \langle \tilde{\mathbf{E}} \otimes \tilde{\mathbf{E}}^* \rangle. \quad (74)$$

The coherent part is indeed known for our problem. Our primary interest is in the diffuse part. Therefore, we write (73) in terms of diffuse fields:

$$\langle \tilde{\mathbf{E}}_j \otimes \tilde{\mathbf{E}}_j^* \rangle = \langle \langle \tilde{\mathbf{E}}_j \rangle_v \otimes \langle \tilde{\mathbf{E}}_j^* \rangle_v \rangle_s + \sum_{k=1}^N \langle \langle \tilde{\mathbf{G}}_{jk} \rangle_v \otimes \langle \tilde{\mathbf{G}}_{jk}^* \rangle_v \rangle_s \langle v_k \otimes v_k^* \rangle \langle \mathbf{E}_k \otimes \mathbf{E}_k^* \rangle_{vs}, \quad (75)$$

where  $\langle \tilde{\mathbf{E}}_j \rangle_v$  is the fluctuating part of  $\langle \mathbf{E}_j \rangle_v$ . Let us now write (75) in more detail as:

$$\begin{aligned} \langle \tilde{\mathbf{E}}_j(\mathbf{r}) \otimes \tilde{\mathbf{E}}_j^*(\mathbf{r}') \rangle &= \langle \langle \tilde{\mathbf{E}}_j(\mathbf{r}) \rangle_v \otimes \langle \tilde{\mathbf{E}}_j^*(\mathbf{r}') \rangle_v \rangle_s + \sum_{k=1}^N \int_{\Omega_k} d\mathbf{r}_1 \int_{\Omega_k} d\mathbf{r}'_1 \\ &\quad \langle \langle \tilde{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}_1) \rangle_v \otimes \langle \tilde{\mathbf{G}}_{jk}^*(\mathbf{r}', \mathbf{r}'_1) \rangle_v \rangle_s |k_k|^4 C_k(\mathbf{r}_1 - \mathbf{r}'_1) \langle \mathbf{E}_k(\mathbf{r}_1) \otimes \mathbf{E}_k^*(\mathbf{r}'_1) \rangle_{vs}. \end{aligned} \quad (76)$$

To obtain the RT equations, we introduce the Wigner transforms of waves and Green's functions as

$$\mathcal{E}_m \left( \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{k} \right) = \int \langle \mathbf{E}_m(\mathbf{r}) \otimes \mathbf{E}_m^*(\mathbf{r}') \rangle e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} d(\mathbf{r} - \mathbf{r}'), \quad (77)$$

$$\begin{aligned} \mathcal{G}_{mn} \left( \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{k} \middle| \frac{\mathbf{r}_1 + \mathbf{r}'_1}{2}, \mathbf{l} \right) &= \int d(\mathbf{r} - \mathbf{r}') \int d(\mathbf{r}_1 - \mathbf{r}'_1) \\ &\quad e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} e^{i\mathbf{l} \cdot (\mathbf{r}_1 - \mathbf{r}'_1)} \langle \langle \tilde{\mathbf{G}}_{mn}(\mathbf{r}, \mathbf{r}_1) \rangle_v \otimes \langle \tilde{\mathbf{G}}_{mn}^*(\mathbf{r}', \mathbf{r}'_1) \rangle_v \rangle_s. \end{aligned} \quad (78)$$

In terms of these transforms (76) becomes

$$\tilde{\mathcal{E}}_m(\mathbf{r}, \mathbf{k}) = \tilde{\mathcal{E}}_m^s(\mathbf{r}, \mathbf{k}) + \frac{1}{(2\pi)^6} \sum_{n=1}^N |k_n|^4 \int_{\Omega_n} d\mathbf{r}' \int \int \mathcal{G}_{mn}(\mathbf{r}, \mathbf{k} | \mathbf{r}', \boldsymbol{\alpha}) \Phi_n(\boldsymbol{\alpha} - \boldsymbol{\beta}) \mathcal{E}_n(\mathbf{r}', \boldsymbol{\beta}) d\boldsymbol{\beta} d\boldsymbol{\alpha}, \quad (79)$$

where  $\Phi_n$  is the spectral density of the volumetric fluctuations in the  $n$ -th layer. We have used the superscript  $s$  in the first term to indicate that this is due to surface scattering as defined by the first term in (76).

As before translational invariance in azimuth implies the following:

$$\mathcal{E}_m(\mathbf{r}, \mathbf{k}) = \mathcal{E}_m(z, \mathbf{k}), \quad (80a)$$

$$\mathcal{G}_{mn}(\mathbf{r}, \mathbf{k} | \mathbf{r}', \mathbf{l}) = \mathcal{G}_{mn}(z, \mathbf{k} | z', \mathbf{l}; \mathbf{r}_\perp - \mathbf{r}'_\perp). \quad (80b)$$

Using these relations in (79) we have

$$\tilde{\mathcal{E}}_m(z, \mathbf{k}) = \tilde{\mathcal{E}}_m^s(z, \mathbf{k}) + \frac{1}{(2\pi)^6} \sum_{n=1}^N |k_n|^4 \int_{z_n}^{z_{n+1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta} \mathcal{G}_{mn}(z, \mathbf{k} | z', \boldsymbol{\alpha}; 0) \Phi_n(\boldsymbol{\alpha} - \boldsymbol{\beta}) \mathcal{E}_n(z', \boldsymbol{\beta}), \quad (81)$$

where  $\mathcal{G}_{mn}(z, \mathbf{k} | z', \boldsymbol{\alpha}; 0)$  is the Fourier transform of  $\mathcal{G}_{mn}(z, \mathbf{k} | z', \boldsymbol{\alpha}; \mathbf{r}_\perp - \mathbf{r}'_\perp)$  w.r.t.  $\mathbf{r}_\perp - \mathbf{r}'_\perp$  evaluated at the origin of the spectral space. To proceed further, we decompose  $\langle \tilde{\mathbf{G}}_{mn} \rangle_v$  into its components,

$$\langle \tilde{\mathbf{G}}_{mn} \rangle_v = \delta_{mn} \bar{\mathbf{G}}_m^o + \bar{\mathbf{G}}_{mn}^{uu} + \bar{\mathbf{G}}_{mn}^{ud} + \bar{\mathbf{G}}_{mn}^{du} + \bar{\mathbf{G}}_{mn}^{dd}, \quad (82)$$

where the first term is the singular part of the Green's function. The superscripts  $u$  and  $d$  indicate up- and down-going elements of the waves. The other components are due to reflections from boundaries. These are formally constructed using the concept of surface scattering operators as follows (Voronovich, 1999),

$$\left\langle G_{mn}^{ab}(\mathbf{r}, \mathbf{r}') \right\rangle_v^{\mu\nu} = \frac{1}{(2\pi)^4} \int \int \{S_{mn}^{ab}(\mathbf{k}_\perp, \mathbf{k}'_\perp)\}^{\mu\nu} e^{i\mathbf{k}_\perp \cdot \mathbf{r} + iaq_m^\mu(\mathbf{k}_\perp)z} e^{-i\mathbf{k}'_\perp \cdot \mathbf{r}' - ibq_n^\nu(\mathbf{k}'_\perp)z'} d\mathbf{k}_\perp d\mathbf{k}'_\perp, \quad (83)$$

where  $\bar{\mathbf{S}}_{mn}^{ab}$  is the surface scattering operator. The superscript and superscript notations have the same meaning as in Section 2.

We recall that  $\mathcal{G}_{mn}$  is the Wigner transform of  $\left\langle \left\langle \bar{\mathbf{G}}_{mn} \right\rangle_v \otimes \left\langle \bar{\mathbf{G}}_{mn}^* \right\rangle_v \right\rangle_s$ . As mentioned before, it is only in the quasi-uniform limit does the Wigner transform of the coherence function lead to the specific intensity of the RT equation. For our layer geometry, the Green's function is nonuniform. However, each of its components given in (82) is quasi-uniform. When we use (82) to perform the Wigner transform we ignore all cross terms. In other words, we make the following approximation,

$$\mathcal{G}_{mn} \simeq \delta_{mn} \mathcal{G}_m^o + \mathcal{G}_{mn}^{uu} + \mathcal{G}_{mn}^{ud} + \mathcal{G}_{mn}^{du} + \mathcal{G}_{mn}^{dd},$$

where  $\mathcal{G}_{mn}^{ab}$  is the Wigner transform of  $\left\langle \left\langle \bar{\mathbf{G}}_{mn}^{ab} \right\rangle_v \otimes \left\langle \bar{\mathbf{G}}_{mn}^{ab*} \right\rangle_v \right\rangle_s$ . Most of the cross terms are nonuniform and may be neglected under the quasi-uniform field assumption. A few cross terms turn out to be quasi-uniform and their inclusion lead to phase matrices that are different from those of the RT system. It turns out that such additional coherence terms become negligible when the layer thickness is of the same order or greater than the mean free path of the corresponding layer. It is under these conditions, the approximate expression for  $\mathcal{G}_{mn}$  given above is good.

With the introduction of this representation for  $\mathcal{G}_{mn}$  in (81), we can trace up- and down-going waves to obtain the following equations for the coherence function:

$$\begin{aligned} \tilde{\mathcal{E}}_m^u(z, \mathbf{k}) &= \tilde{\mathcal{E}}_m^{su}(z, \mathbf{k}) + \frac{1}{(2\pi)^6} |k_m|^4 \int_{z_m}^z dz' \int d\alpha \int d\beta \mathcal{G}_m^>(z, \mathbf{k}|z', \alpha; 0) \Phi_m(\alpha - \beta) \mathcal{E}_m(z', \beta) \\ &\quad + \frac{1}{(2\pi)^6} \sum_{n=1}^N |k_n|^4 \int_{z_n}^{z_{n-1}} dz' \int d\alpha \int d\beta \mathcal{G}_{mn}^{ua}(z, \mathbf{k}|z', \alpha; 0) \Phi_n(\alpha - \beta) \mathcal{E}_n^a(z', \beta) \end{aligned} \quad (84a)$$

$$\begin{aligned} \tilde{\mathcal{E}}_m^d(z, \mathbf{k}) &= \tilde{\mathcal{E}}_m^{sd}(z, \mathbf{k}) + \frac{1}{(2\pi)^6} |k_m|^4 \int_z^{z_{m-1}} dz' \int d\alpha \int d\beta \mathcal{G}_m^<(z, \mathbf{k}|z', \alpha; 0) \Phi_m(\alpha - \beta) \mathcal{E}_m(z', \beta) \\ &\quad + \frac{1}{(2\pi)^6} \sum_{n=1}^N |k_n|^4 \int_{z_n}^{z_{n-1}} dz' \int d\alpha \int d\beta \mathcal{G}_{mn}^{da}(z, \mathbf{k}|z', \alpha; 0) \Phi_n(\alpha - \beta) \mathcal{E}_n^a(z', \beta) \end{aligned} \quad (84b)$$

Note that summation over  $a = \{u, d\}$  is implied in the above equations. The first term in these equations,  $\tilde{\mathcal{E}}^{sa}$ , represents the contribution due exclusively to surface scattering, and has the following form:

$$\begin{aligned} \{\tilde{\mathcal{E}}_m^{sa}(z, \mathbf{k})\}^{\mu\nu} &= 2\pi\delta \left\{ k_z - \frac{1}{2}a \left[ q_m^\mu(\mathbf{k}_\perp) + q_m^{\nu*}(\mathbf{k}_\perp) \right] \right\} \times \\ &\quad \times e^{ia[q_m^\mu - q_m^{\nu*}]z} \left\langle \left\{ \tilde{\Sigma}_m^a \right\}^{\mu\mu'} \left\{ \tilde{\Sigma}_m^{a*} \right\}^{\nu\nu'}(\mathbf{k}_\perp, \mathbf{k}_{\perp i}) \right\rangle_s E_{\mu'i} E_{\nu'i}^*, \end{aligned} \quad (85)$$



where  $\Sigma_m^a$  is the amplitude of the up-going wave in the  $m$ -th layer after volumetric averaging is performed. This means that it is a random function of surface fluctuations. When we substitute (85) and the expressions for  $\mathcal{G}_{mn}$  in (84) we find that

$$\{\tilde{\mathcal{E}}_m^a(z, \mathbf{k})\}_{\mu\nu} = 2\pi\delta \left\{ k_z - \frac{1}{2}a[q_m^\mu(\mathbf{k}_\perp) + q_m^{v*}(\mathbf{k}_\perp)] \right\} e^{ia[q_m^\mu - q_m^{v*}]z} \{\tilde{\mathcal{E}}_m^a(z, \mathbf{k}_\perp)\}_{\mu\nu}. \quad (86)$$

On substituting this in (84) and differentiating w.r.t.  $z$  we obtain the following transport equations:

$$\left\{ \frac{d}{dz} - i[q_\mu(\mathbf{k}_\perp) - q_v^*(\mathbf{k}_\perp)] \right\} \tilde{\mathcal{E}}_{\mu\nu}^u(z, \mathbf{k}_\perp) = \tilde{\mathcal{E}}_{\mu\nu}^u(z, \mathbf{k}_\perp) + \frac{|k_m|^4}{(2\pi)^2} \int d\alpha_\perp S_\mu^> S_v^{>*} \Phi_m \left\{ \mathbf{k}_\perp - \alpha_\perp; \frac{1}{2}[q_\mu(\mathbf{k}_\perp) + q_v^*(\mathbf{k}_\perp)] - \frac{1}{2}a[q_{\mu'}(\alpha_\perp) + q_{v'}^*(\alpha_\perp)] \right\} (\boldsymbol{\mu} \cdot \boldsymbol{\mu}')(\boldsymbol{\nu} \cdot \boldsymbol{\nu}') \tilde{\mathcal{E}}_{\mu'\nu'}^a(z, \alpha_\perp), \quad (87a)$$

$$\left\{ -\frac{d}{dz} - i[q_\mu(\mathbf{k}_\perp) - q_v^*(\mathbf{k}_\perp)] \right\} \tilde{\mathcal{E}}_{\mu\nu}^d(z, \mathbf{k}_\perp) = \tilde{\mathcal{E}}_{\mu\nu}^d(z, \mathbf{k}_\perp) + \frac{|k_m|^4}{(2\pi)^2} \int d\alpha_\perp S_\mu^< S_v^{<*} \Phi_m \left\{ \mathbf{k}_\perp - \alpha_\perp; -\frac{1}{2}[q_\mu(\mathbf{k}_\perp) + q_v^*(\mathbf{k}_\perp)] - \frac{1}{2}a[q_{\mu'}(\alpha_\perp) + q_{v'}^*(\alpha_\perp)] \right\} (\boldsymbol{\mu} \cdot \boldsymbol{\mu}')(\boldsymbol{\nu} \cdot \boldsymbol{\nu}') \tilde{\mathcal{E}}_{\mu'\nu'}^a(z, \alpha_\perp), \quad (87b)$$

where  $\tilde{\mathcal{E}}_{\mu\nu}^a$  represents scattering due to the coherent part of  $\mathcal{E}$ , whereas the integral terms in (87) represent scattering due to the diffuse part of  $\mathcal{E}$ . We may also regard  $\tilde{\mathcal{E}}_{\mu\nu}^a$  as the source to our transport equations; it is given as

$$\begin{aligned} \tilde{\mathcal{E}}_{\mu\nu}^a = & |k_m|^4 \Phi_m \left\{ \mathbf{k}_\perp - \mathbf{k}_{\perp i}; \frac{1}{2}a[q_\mu(\mathbf{k}_\perp) + q_v^*(\mathbf{k}_\perp)] - \frac{1}{2}b[q_\mu(\mathbf{k}_{\perp i}) + q_v^*(\mathbf{k}_{\perp i})] \right\} \times \\ & \times \left( S_\mu^a S_\mu^{a*} \right) (\boldsymbol{\mu} \cdot \boldsymbol{\mu}_i) (\boldsymbol{\nu} \cdot \boldsymbol{\nu}_i) \left( \langle S_{m0}^b \rangle_{\mu i} E_{\mu i} \right) \left( \langle S_{m0}^b \rangle_{\nu i} E_{\nu i} \right)^*, \end{aligned} \quad (88)$$

where summation over  $b$  is implied. Note that  $\tilde{\mathcal{E}}^a$  in (87) includes both  $\tilde{\mathcal{E}}^u$  and  $\tilde{\mathcal{E}}^d$  (corresponding to up- and down-going waves). When the superscripts  $\{a, b\}$  correspond to  $u$ , the value of  $\{a, b\}$  in the argument of  $\Phi_m$  takes the value  $+1$ ; on the other hand, when the superscripts  $\{a, b\}$  correspond to  $d$  the value of  $a$  in the argument of  $\Phi_m$  takes the value  $-1$ . Since all quantities in (87) and (88) correspond to the same layer  $m$  we have dropped the subscript  $m$  to avoid cumbersome notations. Summation over  $\mu'$  and  $\nu'$  is implicit in (87). Similarly, summation over  $\mu_i$  and  $\nu_i$  is implicit in (88).  $S_{m0}^b$  is the scattering amplitude of waves with direction  $b$  in  $m$ -th layer due to wave incident in Region 0. To obtain appropriate boundary conditions we have to go back to the integral equation representations for  $\tilde{\mathcal{E}}_{\mu\nu}^u$  and  $\tilde{\mathcal{E}}_{\mu\nu}^d$  and observe their behaviour at the interfaces and try to find a relation between them.

After some manipulations we arrived at the following boundary conditions. At the  $(m-1)$ -th interface we have

$$\tilde{\mathcal{E}}_m^d(z_{m-1}, \mathbf{k}_\perp) = \int d\mathbf{k}'_\perp \left\langle \ddot{\mathcal{R}}_{m-1,m}(\mathbf{k}_\perp, \mathbf{k}'_\perp) \right\rangle \tilde{\mathcal{E}}_m^u(z_{m-1}, \mathbf{k}'_\perp), \quad (89a)$$

with  $\ddot{\mathcal{R}} = \ddot{\mathbf{R}} \otimes \ddot{\mathbf{R}}^*$  where  $\ddot{\mathbf{R}}_{m-1,m}$  is the stack reflection matrix (not the local reflection matrix) for a wave incident from below on the  $(m-1)$ -th interface. Similarly

$$\tilde{\mathcal{E}}_m^u(z_{m-1}, \mathbf{k}_\perp) = \int d\mathbf{k}'_\perp \left\langle \ddot{\mathcal{R}}_{m+1,m}(\mathbf{k}_\perp, \mathbf{k}'_\perp) \right\rangle \tilde{\mathcal{E}}_m^d(z_{m-1}, \mathbf{k}'_\perp), \quad (89b)$$

where  $\ddot{\mathcal{R}}_{m+1,m}$  is the tensor product of the stack reflection matrix for a wave incident from above on the  $m$ -th interface.

We were able to obtain the boundary conditions only after imposing certain approximations as given below. Consider the following identity:

$$\bar{\mathbf{S}}_{mm}^{du} = \bar{\mathbf{F}}_m \ddot{\mathbf{R}}_{m-1,m} \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \} \bar{\mathbf{F}}_m \quad (90)$$

where  $\bar{\mathbf{F}}_m = \text{diag} \{ e^{iq_h d_m}, e^{iq_v d_m} \}$ . Notice that this is an operator relation where all elements are operators. Taking the tensor product of (90) with its complex conjugate we have

$$\bar{\mathbf{S}}_{mm}^{du} \otimes \bar{\mathbf{S}}_{mm}^{du*} = (\bar{\mathbf{F}}_m \otimes \bar{\mathbf{F}}_m^*) \left( \ddot{\mathbf{R}}_{m-1,m} \otimes \ddot{\mathbf{R}}_{m-1,m}^* \right) \left( \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \} \otimes \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \}^* \right) (\bar{\mathbf{F}}_m \otimes \bar{\mathbf{F}}_m^*). \quad (91)$$

Next we average (91) w.r.t. surface fluctuations and get

$$\left\langle \bar{\mathbf{S}}_{mm}^{du} \otimes \bar{\mathbf{S}}_{mm}^{du*} \right\rangle \simeq (\bar{\mathbf{F}}_m \otimes \bar{\mathbf{F}}_m^*) \left\langle \ddot{\mathbf{R}}_{m-1,m} \otimes \ddot{\mathbf{R}}_{m-1,m}^* \right\rangle \left\langle \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \} \otimes \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \}^* \right\rangle (\bar{\mathbf{F}}_m \otimes \bar{\mathbf{F}}_m^*) \quad (92a)$$

where the two tensor products in the middle are assumed to be weakly correlated. A further approximation that we impose is given as follows

$$\left\langle \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \} \otimes \{ \bar{\mathbf{S}}_m^> + \bar{\mathbf{S}}_{mm}^{uu} \}^* \right\rangle \simeq \bar{\mathbf{S}}_m^> \otimes \bar{\mathbf{S}}_m^{>*} + \left\langle \bar{\mathbf{S}}_{mm}^{uu} \otimes \bar{\mathbf{S}}_{mm}^{uu*} \right\rangle. \quad (92b)$$

This is similar to the approximation we used while computing the Wigner transforms of the Green's functions. These are the kinds of approximations required to arrive at our boundary conditions.

### 3.4 Transition to radiative transfer

The procedure for transition to radiative transfer is identical to the planar interface problem and hence we do not repeat it here. We find that the conditions necessary to connect the RTT with statistical wave theory are:

- (a) layer thickness is greater than the mean free path of the corresponding medium,
- (b) quasi-homogeneous field approximation.

By following the same procedure as in Section 2 we obtain the transport theoretic boundary conditions from (89).

### 3.5 Remarks

Now that we have made the transition from statistical wave theory to radiative transfer theory, we itemize the assumptions implicitly involved in the RT approach.

1. Quasi-stationary field approximation.
2. Weak fluctuations.
3. Statistical homogeneity of fluctuations.

These are the three well-known conditions necessary for the unbounded random media problem. However, if the medium is bounded with rough interfaces we need to impose additional conditions. We found that the extinction coefficients calculated in the statistical wave approach and the RT approach are different and only after applying further approximations can they be made to agree with each other. The following additional

conditions are required for our bounded random medium problem:

4. Layer thickness must be of the same order or greater than the corresponding mean free path.
5. Weak surface correlation approximation.
6. All fluctuations of the problem are statistically independent.

#### 4. Conclusion

To summarize, we have enquired into the assumptions involved in adopting the radiative transfer approach to scattering from layered random media with rough interfaces. To facilitate this enquiry we adopted a statistical wave approach to this problem and derived the governing equations for the first and second moments of the wave fields. We employed Wigner transforms and transitioned to the system corresponding to that of radiative transfer approach. In this process we found that there are more conditions implicitly involved in the RT approach to this problem than it is widely believed to be sufficient. With the recent development of fast and efficient algorithms for scattering computations and the enormous increase in computer resources it is now feasible to take an entirely numerical approach to this problem without imposing any approximations. In spite of such developments, to keep the size of the problem manageable only special cases have been studied thus far (Giovannini et al., 1998; Peloci & Coccioli, 1997; Pak et al., 1993; Sarabandi et al., 1996). Hence it is very much of relevance, interest, and convenience to apply the RT approach to these problems. However, one should keep in mind the assumptions involved in such an approach. Otherwise interpretations of results based on RT theory can be misleading.

In this work we have modelled the random media as random continua. Another approach to this problem is the discrete random medium model (Foldy, 1945; Lax, 1951; Twersky, 1964; Ishimaru, 1997; Tsang et al., 1985; Mishchenko et al., 2006). Recently Mishchenko (Mishchenko, 2002) (hereafter referred to as MTL for brevity) derived the vector radiative transfer equation (VRTE) for a bounded discrete random medium using a rigorous microphysical approach. This enabled them to identify the following assumptions embedded in the VRTE.

1. Scattering medium is illuminated by a plane wave.
2. Each particle is located in the far-field zone of all other particles and the observation point is also located in the far-field zones of all the particles forming the scattering medium.
3. Neglect all scattering paths going through a particle two or more times (Twersky approximation).
4. Assume that the scattering system is ergodic and averaging over time can be replaced by averaging over particle positions and states.
5. Assume that (i) the position and state of each particle are statistically independent of each other and of those of all other particles and (ii) the spatial distribution of the particles throughout the medium is random and statistically uniform.
6. Assume that the scattering medium is convex.
7. Assume that the number of particles  $N$  forming the scattering medium is very large.
8. Ignore all the diagrams with crossing connections in the diagrammatic expansion of the coherency dyadic.

It is apparent that there are distinct differences in the analyses for scattering from discrete and continuous random media. Hence it is not possible to make a one-to-one correspondence between the conditions of MTL and those in this work. Below is an attempt to make a connection between the two by considering each condition derived by MTL and relating it

to ours. We will denote the condition numbers derived by MTL as MTL # and those obtained in this work as SM #.

**MTL 1:-** We also have a plane electromagnetic wave illuminating our system, although as pointed out by MTL it can be a quasi-plane wave.

**MTL 2:-** We also have implicitly employed the far-field approximation. It is embedded in SM 1.

**MTL 3:-** This is embedded in SM 2. Although not explicitly stated, the scattering processes as mentioned in MTL 3 are avoided.

**MTL 4:-** In this work we have restricted our attention to the time-independent problem and hence did not encounter the issue of ergodicity.

**MTL 5:-** This condition is embedded in SM 3.

**MTL 6:-** In our problem we have distinct scattering boundaries and the character of the waves exiting or entering them are explicitly contained in the boundary conditions. Hence convexity of the scattering medium is not a necessary condition for us.

**MTL 7:-** This condition is embedded in SM 4.

**MTL 8:-** This condition is embedded in SM 2. Under weak fluctuation approximation we only take into consideration the leading term of the intensity operator.

Since the problem that we considered in this work involved scattering boundaries we have some additional conditions beyond those of MTL. Although the main conclusions obtained are the same for the problem with random continuum and discrete random medium there are some peculiarities with the discrete random medium case and hence there are some differences in the assumptions implied in the RT approach.

There are a few more remarks that we would like to make before closing.

(a) In RT theory the medium is assumed to be sparse and hence the “refraction effects” of the fluctuations are ignored. Thus in the boundary conditions we should use the background medium parameters rather than the effective medium parameters as derived in our statistical wave theory.

(b) To arrive at (46) from (45) we have ignored the contribution of evanescent modes.

(c) The condition about statistical homogeneity of fluctuations may be relaxed by assuming it to be statistically quasi-homogeneous and we still can arrive at our results without much difficulty.

(d) The assumption regarding the underlying statistics to be Gaussian is not only a convenience but also a reasonably good approximation in many applications. However, there are indeed certain situations where the statistics are not Gaussian. Similar analysis for such more general statistics are more complex and involved.

## 5. Acknowledgment

The author thanks the US Air Force of Scientific Research for support.

## 6. References

- Ambartsumian, V. A. (1943). Diffuse scattering of light by a turbid medium, *Dokl. Akad. Nauk SSSR* 38: 85–91.
- Amra, C. (1994). Light scattering from multilayer optics I. Tools of investigation, *J. Opt. Soc. Am. A* 11: 197–210.
- Arridge, S. R. & Hebden, J. C. (1997). Optical imaging in medicine: II. Modelling and reconstruction, *Phys. Med. Biol.* 42: 841–853.

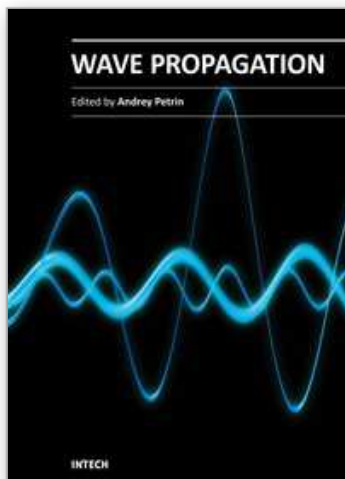
- Barabanenkov, Y. N., Vinogradov, A. G., Kravtsov, Y. A. & Tatarskii, V. I. (1972). Application of the theory of multiple scattering of waves to the derivation of the radiation transfer equation for a statistically inhomogeneous medium, *Izv. VUZ Radiofiz.* 15: 1852–1860.
- Bass, F. G. & Fuks, I. M. (1979). *Wave Scattering from Statistically Rough Surfaces*, Pergamon, Oxford.
- Beckmann, P. & Spizzichino, A. (1987). *The Scattering of Electromagnetic Waves from Rough Surfaces*, Artech House, Norwood, Massachusetts.
- Berk, A., Bernstein, L. S., G. P. Anderson, P. K. A., Robertson, D. C., Chetwynd, J. H. & Adler-Golden, S. H. (1998). MODTRAN cloud and multiple scattering upgrades with application to AVIRIS, *Remote Sensing of Environment* 65: 367–375.
- Brown, G. S. (1988). A theoretical study of the effects of vegetation on terrain scattering, *Technical Report RADC-TR-89-64*, Rome Air Development Center, Rome, NY.
- Caron, J., Andraud, C. & Lafait, J. (2004). Radiative transfer calculations in multilayer system with smooth or rough interfaces, *J. Mod. Opt.* 51: 575–595.
- Chandrasekar, S. (1960). *Radiative Transfer*, Dover, New York.
- Chauhan, N. S., Lang, R. H. & Ranson, K. J. (1991). Radar modelling of boreal forest, *IEEE Trans. Geosci. Rem. Sens.* 29: 627–638.
- Chew, W. (1995). *Waves and Fields in Inhomogeneous Media*, IEEE Press, Piscataway, New Jersey.
- Clough, S. A., Shephard, M. W., Milawer, E. J., Delamere, J. S., Iacono, M. J., Cady-Pereira, K., Boukabara, S. & Brown, P. D. (2005). Atmospheric radiative transfer modeling: A summary of the AER codes, *J. Quant. Spectrosc. Radiat. Transfer* 91: 233–244.
- Daniels, D. J. (2004). *Ground Penetrating Radar*, 2nd edn, IEE, Herts, United Kingdom.
- de Hulst, H. C. V. (1980). *Multiple Light Scattering: Tables, Formulas, and Applications*, Academic Press, London, United Kingdom.
- Elachi, C. & van Zyl, J. J. (2006). *Introduction to the Physics and Techniques of Remote Sensing*, 2nd edn, John Wiley and Sons, Hoboken, New Jersey.
- Elson, J. M. (1995). Multilayer-coated optics: Guided-wave coupling and scattering by means of interface roughness, *J. Opt. Soc. Am. A* 12: 729–742.
- Elson, J. M. (1997). Characteristics of far field scattering by means of surface roughness and variations in subsurface permittivity, *Waves Random Media* 7: 303–317.
- Foldy, L. L. (1945). The multiple scattering of waves, *Phys. Rev.* 67: 107–117.
- Friberg, A. T. (1986). Energy transport in optical systems with partially coherent light, *Applied Optics* 25: 4547–4556.
- Fung, A. K. & Chen, M. (1981). Scattering from a Rayleigh layer with an irregular interface, *Radio Sci.* 16: 1337–1347.
- Giovannini, H., Saillard, M. & Sentenac, A. (1998). Numerical study of scattering from rough inhomogeneous films, *J. Opt. Soc. Am. A* 15: 1182–1191.
- Hasekamp, O. P., Landgraf, J. & van Oss, R. (2002). The need of polarization modelling for ozone profile retrieval from backscattered sunlight, *J. Geophys. Res.* 107: 4692.
- Ishimaru, A. (1997). *Wave Propagation and Scattering in Random Media*, IEEE Press, New York.
- Karam, M. A. & Fung, A. K. (1982). Propagation and scattering in multi-layer random media with rough interfaces, *Electromagnetics* 2: 239–256.
- Kattawar, G. W. & Adams, C. N. (1990). Errors in radiance calculations induced by using scalar rather than Stokes vector theory in realistic atmosphere-ocean system, in R. W. Spinrad (ed.), *Ocean Optics X*, Vol. 1302 of *Proc. of SPIE*, pp. 2–12.
- Kuo, C. H. & Moghaddam, M. (2007). Electromagnetic scattering from multilayer rough surfaces with arbitrary dielectric profiles for remote sensing of subsurface soil



- moisture, *IEEE Trans. Geosci. Rem. Sens.* 45: 349–366.
- Lam, C. M. & Ishimaru, A. (1993). Muller matrix representation for a slab of random medium with discrete particles and random rough surfaces, *Waves Random Media* 3: 111–125.
- Lax, M. (1951). Multiple scattering of waves, *Rev. Mod. Phys.* 23: 287–310.
- Lee, J. K. & Kong, J. A. (1985). Active microwave remote sensing of an anisotropic layered random medium, *IEEE Trans. Geosci. Rem. Sens.* 23: 910–923.
- Lenoble, J. (1993). *Atmospheric Radiative Transfer*, A. Deepak Publishing, Hampton, Virginia.
- Lenoble, J. (ed.) (1985). *Radiative Transfer in Scattering and Absorbing Atmospheres: Standard Computational Procedures*, A. Deepak Publishing, Hampton, Virginia.
- Levy, R. C., Remer, L. A. & Kaufman, Y. J. (2004). Effects of neglecting polarization on the MODIS aerosol retrieval over land, *IEEE Trans. Geosci. Rem. Sens.* 42: 2576–2583.
- Liang, P., Moghaddam, M., Pierce, L. E. & Lucas, R. M. (2005). Radar backscattering model for multilayer mixed species, *IEEE Trans. Geosci. Rem. Sens.* 43: 2612–2626.
- Marchand, E. W. & Wolf, E. (1974). Radiometry with sources in any state of coherence, *J. Opt. Soc. Am.* 64: 1219–1226.
- Mishchenko, M. I. (2002). Vector radiative transfer equation for arbitrarily shaped and arbitrarily oriented particles: A microphysical derivation from statistical electrodynamics, *Appl. Opt.* 41: 7114–7134.
- Mishchenko, M. I., Travis, L. D. & Lacis, A. A. (2006). *Multiple Scattering of Light by Particles*, Cambridge University Press, New York.
- Moghaddam, M., Rahmat-Samii, Y., E.Rodriguez, Entekhabi, D., Hoffman, J., Moller, D., Pierce, L. E., Saatchi, S. & Thomson, M. (2007). Microwave Observatory of Subcanopy and Subsurface (MOSS): A mission concept of global deep soil moisture observations, *IEEE Trans. Geosci. Remote Sens.* 45: 2630–2643.
- Mudaliar, S. (1994). Electromagnetic wave scattering from a random medium layer with a rough interface, *Waves Random Media* 4: 167–176.
- Mudaliar, S. (2005). Statistical wave theory for a random medium layer over a two-scale rough surface, *Proc. IASTED Intl. Conf. Antennas, Radars and Wave Propagation*, pp. 246–253.
- Pak, K., Tsang, L., Li, L. & Chan, C. H. (1993). Combined random rough surface and volume scattering based on Monte-Carlo simulation of Maxwell equations, *Radio Sci.* 23: 331–338.
- Papa, R. J. & Tamaskanis, D. (1991). A model for scattering of electromagnetic waves from foliage-covered terrain, Vol. 46 of *AGARD Conf. Proc.*, pp. 4.1–4.12.
- Pederson, H. M. & Stamnes, J. J. (2000). Radiometric theory of spatial coherence in free space propagation, *J. Opt. Soc. Am. A* 17: 1413–1420.
- Peloci, G. & Coccioli, K. (1997). A finite element approach for scattering from inhomogeneous media with a rough interface, *Waves Random Media* 7: 119–127.
- Sarabandi, K., Yisok, O. & Ulaby, F. (1996). A numerical simulation of scattering from one-dimensional inhomogeneous dielectric rough surfaces, *IEEE Trans. Geosci. Remote Sens.* 34: 425–432.
- Sato, H. & Fehler, M. C. (1998). *Seismic Wave Propagation and Scattering in the Heterogeneous Earth*, Springer-Verlag, New York.
- Shin, R. T. & Kong, J. A. (1989). Radiative transfer theory for active remote sensing of two layer random medium, *PIER* 3: 359–419.
- Sobolev, V. V. (1963). *A Treatise on Radiative Transfer*, Van Nostrand, Princeton, New Jersey.
- Stam, D. M. & Hovenier, J. W. (2005). Errors in calculated planetary phase functions and



- albedos due to neglecting polarization, *Astron. Astrophys.* 444: 275–286.
- Stammes, P. (1994). Errors in UV reflectivity and albedo calculations due to neglecting polarization, in R. P. Santer (ed.), *Atmospheric Gases*, Vol. 2311 of *Proc. of SPIE*, pp. 227–235.
- Stamnes, K., Tsay, S. C., Wiscombe, W. & Jayweera, K. (1988). Numerically stable algorithm for discrete-ordinate-method radiative transfer in multiple scattering and emitting layered media, *Appl. Opt.* 27: 2502–2509.
- Tatarskii, V. I. (1971). *The Effects of Turbulent Atmosphere on Wave Propagation*, Israel Program of Scientific Translations, Jerusalem, Israel.
- Tsang, L., Kong, J. A. & Shin, R. T. (1985). *Theory of Microwave Remote Sensing*, John Wiley, New York.
- Twersky, V. (1964). On propagation in random media of discrete scatterers, Vol. 16 of *Proc. of the Symposium of Applied Mathematics*, pp. 84–116.
- Twersky, V. (1980). Multiple scattering of waves by correlated distributions, in J. A. DeSanto, A. W. Saenz & W. W. Zachary (eds), *Mathematical Methods and Applications in Scattering Theory*, Vol. 130 of *Lecture Notes in Physics*, Springer Verlag, New York, USA, pp. 1–9.
- Ulaby, F. T., Moore, R. K. & Fung, A. K. (1986). *Microwave Remote Sensing: Active and Passive*, Vol. 3, Artech House, Norwood, Massachusetts.
- Ulaby, F. T., Sarabandi, K., McDonald, K., Whitt, K. M. & Dobson, M. C. (1990). Michigan microwave canopy model, *Int. J. Rem. Sens.* 11: 1223–1253.
- Urbini, S., Vitterari, L. & Gandolfi, S. (2001). GPR and GPS data integration: example of application in Antarctica, *Annali di Geofisica* 44: 687–701.
- Voronovich, A. G. (1999). *Wave Scattering from Rough Surfaces*, 2nd edn, Springer-Verlag, Berlin.
- Yoshimori, K. (1998). Radiometry and coherence in a nonstationary optical field, *J. Opt. Soc. Am. A* 15: 2730–2734.
- Zuniga, M. A., Habashy, T. M. & Kong, J. A. (1979). Active remote sensing of layered random media, *IEEE Trans. Geosci. Rem. Sens.* 17: 296–302.



## **Wave Propagation**

Edited by Dr. Andrey Petrin

ISBN 978-953-307-275-3

Hard cover, 570 pages

**Publisher** InTech

**Published online** 16, March, 2011

**Published in print edition** March, 2011

The book collects original and innovative research studies of the experienced and actively working scientists in the field of wave propagation which produced new methods in this area of research and obtained new and important results. Every chapter of this book is the result of the authors achieved in the particular field of research. The themes of the studies vary from investigation on modern applications such as metamaterials, photonic crystals and nanofocusing of light to the traditional engineering applications of electrodynamics such as antennas, waveguides and radar investigations.

### **How to reference**

In order to correctly reference this scholarly work, feel free to copy and paste the following:

Saba Mudaliar (2011). Radiative Transfer Theory for Layered Random Media, Wave Propagation, Dr. Andrey Petrin (Ed.), ISBN: 978-953-307-275-3, InTech, Available from: <http://www.intechopen.com/books/wave-propagation/radiative-transfer-theory-for-layered-random-media>

**INTECH**  
open science | open minds

### **InTech Europe**

University Campus STeP Ri  
Slavka Krautzeka 83/A  
51000 Rijeka, Croatia  
Phone: +385 (51) 770 447  
Fax: +385 (51) 686 166  
[www.intechopen.com](http://www.intechopen.com)

### **InTech China**

Unit 405, Office Block, Hotel Equatorial Shanghai  
No.65, Yan An Road (West), Shanghai, 200040, China  
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元  
Phone: +86-21-62489820  
Fax: +86-21-62489821

© 2011 The Author(s). Licensee IntechOpen. This chapter is distributed under the terms of the [Creative Commons Attribution-NonCommercial-ShareAlike-3.0 License](https://creativecommons.org/licenses/by-nc-sa/3.0/), which permits use, distribution and reproduction for non-commercial purposes, provided the original is properly cited and derivative works building on this content are distributed under the same license.

IntechOpen

IntechOpen