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Recent Progress in Synchronization of Multiple Time Delay Systems

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1. Introduction

The phenomenon of synchronization of dynamical systems was reported by the famous Dutch scientist Christiaan Huygens in 1665 on his observation of synchronization of two pendulum clocks. And, chaos theory has been aroused and developed very early (since 1960s) with efforts in many different research fields, such as mathematics (Li & Yorke, 1975; Ruelle, 1980; Sharkovskii, 1964; 1995), physics (Feigenbaum, 1978; Hénon, 1976; Rossler, 1976), chemistry (Zaikin & Zhabotinsky, 1970; Zhabotinsky, 1964), biology (May, 1976) and engineering (Lorenz, 1963a;b; Nakagawa, 1999), etc (Gleick, 1987; Stewart, 1990). However, until 1983, the idea of synchronization of chaotic systems was raised by Fujisaka and Yamada (Fujisaka & Yamada, 1983). There, the general stability theory of the synchronized motions of the coupled-oscillator systems with the use of the extended Lyapunov matrix approach, and the coupled Lorenz model was investigated as an typical example. A typical synchronous system can be seen in Fig.1. In 1990, Pecora and Carroll (Pecora & Carroll, 1990) realized chaos synchronization in the form of drive-response under the identical synchronous scheme. Since then, chaos synchronization has been aroused and it has become the subject of active research, mainly due to its potential applications in several engineering fields such as communications (Kocarev et al., 1992; Parlitz et al., 1992; Parlitz, Kocarev, Stojanovski & Preckel, 1996), lasers (Fabiny et al., 1993; Roy & Thornburg, 1994), ecology (Blasius et al., 1999), biological systems (Han et al., 1995), system identification (Parlitz, Junge & Kocarev, 1996), etc. The research evolution on chaos synchronization has led to several schemes of chaos synchronization proposed successively and pursued, i.e., generalized (Rulkov et al., 1995), phase (Rosenblum et al., 1996), lag (Rosenblum et al., 1997), projective (Mainieri & Rehacek, 1999), and anticipating (Voss, 2000) synchronizations. Roughly speaking, synchronization of coupled dynamical systems can be interpreted to mean that the master sends the driving signal to drive the slave, and there exists some functional relations in their trajectories during interaction. In fact, the difference between synchronous schemes is lied in the difference of functional relations in trajectories. In other words, a certain functional relation expresses the particular characteristic of corresponding synchronous scheme. When a synchronous regime is established, the expected functional relation is achieved and *synchronization manifold* is usually used to refer to such specific relation in a certain coupled systems.

Time delay systems have been studied in both theory (Krasovskii, 1963) and application (Loiseau et al., 2009). The prominent feature of chaotic time-delay systems is that they have very complicated dynamics (Farmer, 1982). Analytical investigation on time-delay systems by Farmer has showed that it is very easy to generate chaotic behavior even in systems with a single equation with a single delay such as Mackey-Glass's, Ikeda's. Recently, researchers have been attracted by synchronization issues in coupled time-delay systems. Accordingly, several synchronous schemes have been proposed and pursued. However, up-to-date research works have been restricted to the synchronization models of single-delay (Pyragas, 1998a; Senthilkumar & Lakshmanan, 2005) and multiple time delay systems (MTDSs) (Shahverdiev, 2004; Shahverdiev et al., 2005; Shahverdiev & Shore, 2005). There, coupling (or driving) signals are in the form of either linear or single nonlinear transform of state variable. Those models of synchronization in coupled time-delay systems can be used in secure communications (Pyragas, 1998b), however, the security is not assured (Ponomarenko & Prokhorov, 2002; Zhou & Lai, 1999) due that there are several advanced reconstruction techniques which can infer the system's dynamics. From such the fact, synchronization of MTDSs has been intensively investigated (Hoang et al., 2005). In this chapter, recent development for synchronization in coupled MTDSs has been reported. The examples will illustrate the existence and transition in various synchronous schemes in coupled MTDSs.

The remainder of the chapter is organized as follows. Section 2 introduces the MTDSs and its complexity. The proposed synchronization models of coupled MTDSs with various synchronous schemes are described in Section 3. Numerical simulation for proposed synchronization models is illustrated in Section 4. The discussions and conclusions for the proposed models are given in the last two sections.

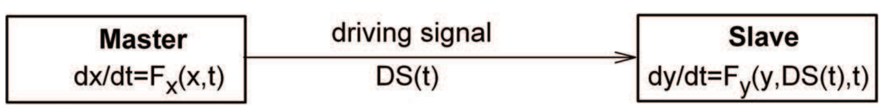


Fig. 1. A typical synchronous system.

2. Multiple time-delay systems

2.1 Overview of time-delay feedback systems

Let us consider the equation representing for a single time-delay system (STDS) as below

$$\frac{dx}{dt} = -\alpha x + f(x(t-\tau)) \tag{1}$$

where α and τ are positive real numbers, τ is a time length of delay applied to the state variable. $f(x) = \frac{x}{1+x^{10}}$ and $f(x) = sin(x)$ are well-known time-delay feedback systems; Mackey-Glass (Mackey & Glass, 1977) and Ikeda (Ikeda & Matsumoto, 1987) systems, respectively. α and/or τ can be used for controlling the complexity of chaotic dynamics (Farmer, 1982). An analog circuit model (Namajūnas et al., 1995) of STDSs is depicted in Fig. 2. The dynamical model of the circuit can be written as

$$\frac{dU}{dt} = \frac{U_{ND}(t) - U(t)}{C_0 R_0} \tag{2}$$

where $U_{ND}(t) = f(U(t-\tau))$. Apparently, the equations given in Eqs. (1) and (2) has the same form.

Chaos synchronization of coupled STDSs has been studied and experimented in several

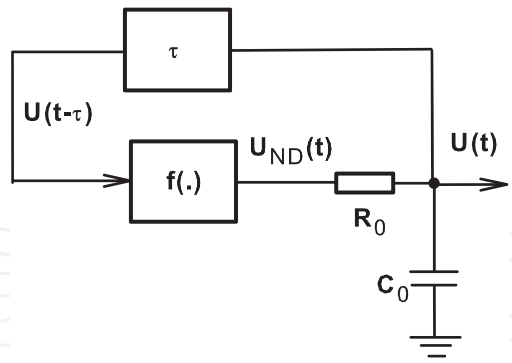


Fig. 2. Circuit model of single delay feedback systems

fields such as circuits (Kim et al., 2006; Kittel et al., 1998; Namajūnas et al., 1995; Sano et al., 2007; Voss, 2002), lasers (Celka, 1995; Goedgebuer et al., 1998; Lee et al., 2006; Masoller, 2001; S. Sivaprakasam et al., 2002; Zhu & WU, 2004), etc. So far, most of research works in this context have focused on synchronization models of STDSs (Pyragas, 1998a; Senthilkumar & Lakshmanan, 2005), in which Mackey-Glass (Mackey & Glass, 1977) and Ikeda (Ikeda & Matsumoto, 1987) systems have been employed as dynamical equations for specific examples. Up to date, there have been several coupling methods for synchronization models of STDSs, i.e., linear (Mensour & Longtin, 1998; Pyragas, 1998a) and single nonlinear coupling (Shahverdiev & Shore, 2005). In other words, the form of driving signals is either $x(t)$ or $f(x(t-\tau))$.

Recently, MTDSs have been interested and aroused (Shahverdiev, 2004). That is because of their potential applications in various fields. A general equation representing for MTDSs is as

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P} m_i f(x(t-\tau_i)) \tag{3}$$

where $m_i, \tau_i \in (\tau_i \geq 0)\Re$. It is clear that MTDSs can been seen as an extension of STDSs. STDSs, MTDSs exhibit chaos.

Chaos synchronization models of MTDSs has been aroused by Shahverdiev *et al.* (Shahverdiev, 2004; Shahverdiev et al., 2005). So far, the studies are constrained to the cases that the coupling (or driving) signal is in the form of linear ($x(t)$) or single nonlinear transform of delayed state variable ($f(x(t-\tau))$). A synchronization model using STDSs with linear form of driving signal can be expressed by

Master:

$$\frac{dx}{dt} = -\alpha x + f(x(t-\tau)) \tag{4}$$

Driving signal:

$$DS(t) = kx \tag{5}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + f(y(t-\tau)) + DS(t) \tag{6}$$

where k is coupling strength. A synchronization model using MTDSs with linear form of driving signal can be expressed by

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P} m_i f(x(t-\tau_i)) \tag{7}$$

Driving signal:

$$DS(t) = kx \tag{8}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P} n_i f(y(t-\tau_i)) + DS(t) \tag{9}$$

In the synchronous system given in Eqs. (7)-(9), if the driving signal is in the form of $DS(t) = kf(x(t-\tau))$, the synchronous system becomes synchronization of MTDSs with single nonlinear driving signal.
In theoretical, such above synchronization models do not offer advantages for the secure communication application due to the fact that their dynamics can be inferred easily by using conventional reconstruction methods (Prokhorov et al., 2005; Voss & Kurths, 1997). By such the reason, seeking for a non-reconstructed time-delay system is important for the chaotic secure communication application. One of the disadvantages of state-of-the-art reconstruction methods is that MTDSs can not be reconstructed if the measured time series is sum of multiple nonlinear transforms of delayed state variable, i.e. $\sum_j f(x(t-\tau_j))$. This is the key hint for proposing a new synchronization model of MTDSs. In the next section, the synchronization models of coupled MTDSs are investigated, in which the driving signal is sum of nonlinearly transformed components of delayed state variable, $\sum_j f(x(t-\tau_j))$. The conditions for synchronization in particular synchronous schemes are considered and proved under the Krasovskii-Lyapunov theory. The numerical simulation will demonstrate and verify the prediction in these contexts.

2.2 The complexity analysis for MTDSs

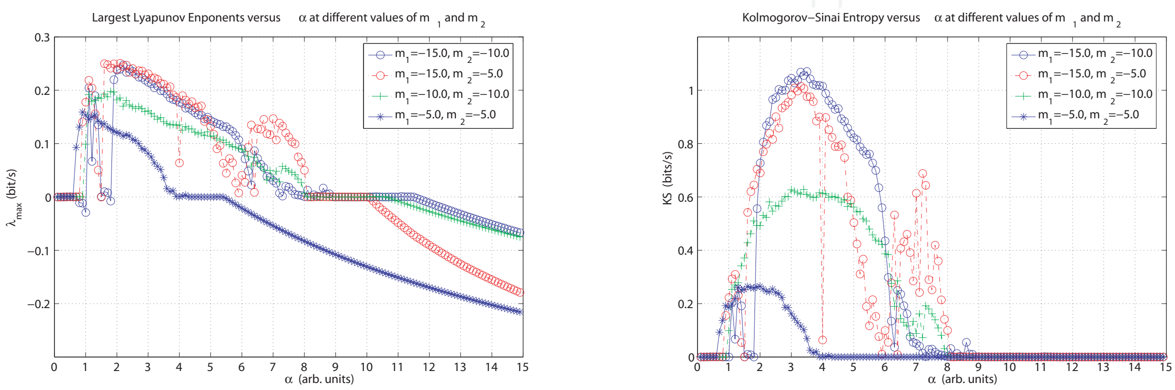
The complexity degree of MTDSs is confirmed that MTDSs not only exhibit hyperchaos, but also bring much more complicated dynamics in comparison with that in single delay systems. This will emphasis significances of MTDSs to the secure communication application. In order to illustrate the complicated dynamics of MTDSs, the Lyapunov spectrum and metric entropy are calculated. Lyapunov spectrum shows the complexity measure while metric entropy presents the predictability to chaotic systems. Here, Kolmogorov-Sinai entropy (Cornfeld et al., 1982) is estimated with

$$KS = \sum_i \lambda_i \quad for\ \lambda_i > 0 \tag{10}$$

The two-delays Mackey-Glass system given in Eq. (11) is studied for this purpose. The complexity degree with respect to values of parameters and of delays is shown by varying α, m_i and τ_i. There are several algorithms to calculate the Lyapunov exponents of dynamical systems as presented in (Christiansen & Rugh, 1997; Grassberger & Procaccia, 1983; Sano & Sawada, 1985; Zeng et al., 1991) and others. However, so far , all of the existing algorithms are inappropriate to deal with the case of MTDSs. Here, estimation of Lyapunov spectrum is based on the algorithm proposed by Masahiro Nakagawa (Nakagawa, 2007)

$$\frac{dx}{dt} = -\alpha x + m_1 \frac{x_{\tau_1}}{1+x_{\tau_1}^{10}} + m_2 \frac{x_{\tau_2}}{1+x_{\tau_2}^{10}} \tag{11}$$

Shown in Fig. 3(a) is largest Lyapunov exponents (LLE) and Kolmogorov-Sinai entropy with some couples of value of m_1 and m_2. In this case, the value of τ_1 and τ_2 is set at 2.5 and 5.0, respectively. The chaotic behavior exhibits in the specific value range of α. Moreover, the range seems to be wider with the increase in the value of $|m_1| + |m_2|$. It is clear to be seen from Fig.3 that the possible largest LEs and metric entropy in this system (approximately 0.3 for largest LEs and 1.4 for metric entropy) are larger in comparison with those of the single delay Mackey-Glass system studied by J.D. Farmer (Farmer, 1982) (approximately 0.07 for largest LEs and 0.1 for metric entropy). It means that the chaotic dynamics of MTDSs is much more complicated than that of single delay systems. As a result, it is hard to reconstruct and predict the motion of MTDSs.



(a) LLEs versus α

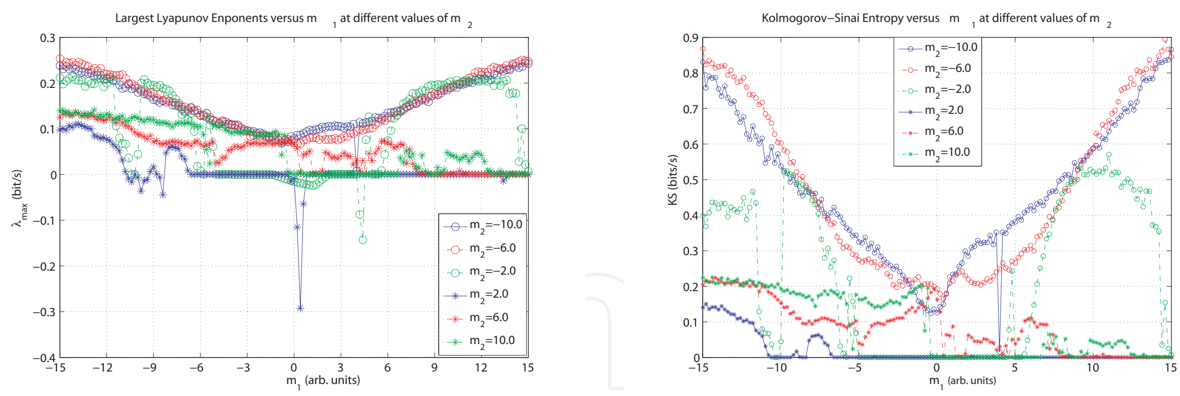
(b) Kolmogorov-Sinai entropy versus α

Fig. 3. Largest LEs and Kolmogorov-Sinai entropy versus α of the two delays Mackey-Glass system.

Illustrated in Figs. 4 and 5 is the largest LEs as well as Kolmogorov-Sinai entropy with respect to the value of m_1 and m_2. There, the value of α, τ_1 and τ_2 is kept constant at 2.1, 2.5 and 5.0, respectively. Noticeably, the two-delays system is with weak feedback in the range of small value of m_1, or the two-feedbacks system tends to single feedback one. In such the range, the curves of largest LEs and Kolmogorov-Sinai entropy are in 'V' shape for negative value of m_2 as depicted in Figs. 4(a) and 4(b). This is also observed in the curves of metric entropy in Fig. 4(b). In other words, the dynamics of MTDSs are intuitively more complicated than that of STDSs. By observing the curves in Figs. 5(a) and 5(b), these characteristics in the case of changing the value of m_2 are a bit different. The range of m_2 offers the 'V' shape is around 3.0 for large negative values of m_1, i.e., -14.5 and -9.5. It can be interpreted that this characteristic depends on the value of delays associated with m_i. As a particular case, the result shows that the trend of largest LEs and metric entropy depends on the value of m_1 and m_2.

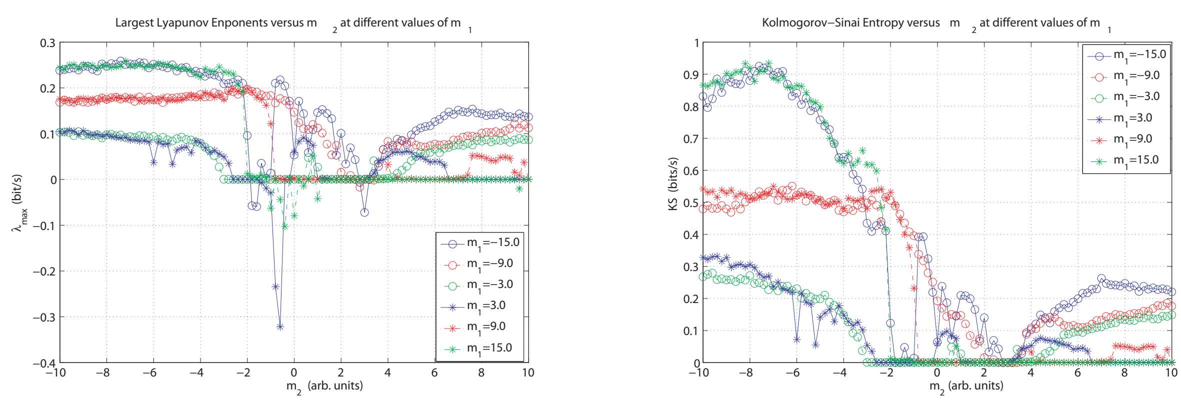
In Fig. 6, the largest LEs and Kolmogorov-Sinai entropy related to the value of τ_1 and τ_2 are presented, and it is clear that they strongly depend on the value of τ_1 and τ_2. There, the value of other parameters is set at $m_1 = -15.0$, $m_2 = -10.0$ and $\alpha = 2.1$. The system still exhibits chaotic dynamics even though the dependence of largest LEs and metric entropy on the value of τ_1 and τ_2 is observed.

In summary, the dynamics of MTDSs is firmly more complicated than that of STDSs. In other words, MTDSs present significances to the secure communication application.



(a) LLEs versus m_1 (b) Kolmogorov-Sinai entropy versus m_1

Fig. 4. Largest LEs and Kolmogorov-Sinai entropy versus m_1 of the two-delays Mackey-Glass system.



(a) LLEs versus m_2 (b) Kolmogorov-Sinai entropy versus m_2

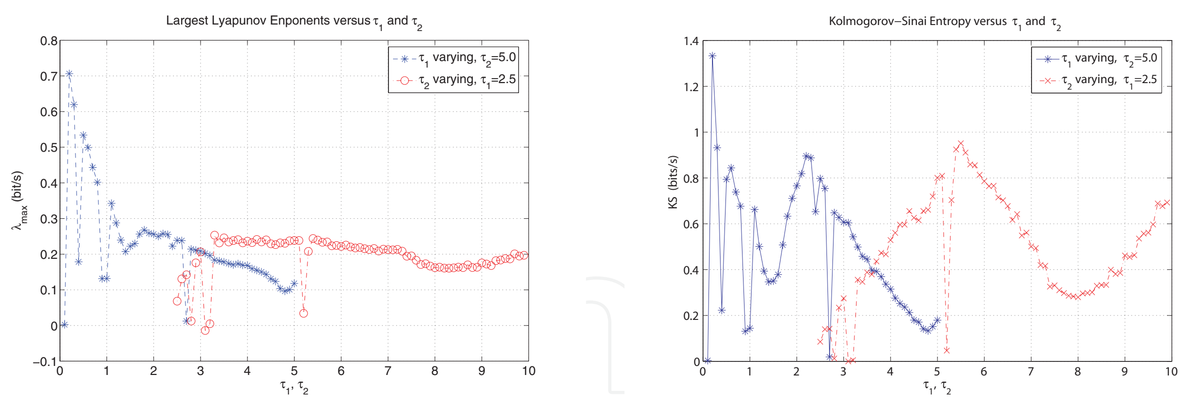
Fig. 5. Largest LEs and Kolmogorov-Sinai entropy versus m_2 of the two-delays Mackey-Glass system.

3. The proposed synchronization models of coupled MTDSs

We consider synchronization models of coupled MTDSs with restriction to the only one state variable. In addition, various schemes of synchronization are investigated on such the synchronization models. The main differences between these proposed models and conventional ones are that dynamical equations for the master and slave are in the form of multiple time delays and the driving signal is constituted by sum of nonlinear transforms of delayed state variable. The condition for synchronization is still based on the Krasovskii-Lyapunov theory. Proofs of the sufficient condition for considered synchronous schemes will also be shown.

3.1 Synchronization of coupled identical MTDSs

We start considering the synchronization of MTDSs with the dynamical equations in the form of one dimension defined by



(a) LLEs versus τ_1 and τ_2 (arb. units)

(b) Kolmogorov-Sinai entropy versus τ_1 and τ_2 (arb. units)

Fig. 6. Largest LEs and Kolmogorov-Sinai entropy versus τ_1 and τ_2 of the two-delays Mackey-Glass system.

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P} m_i f(x_{\tau_i}) \tag{12}$$

Driving signal:

$$DS(t) = \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}}) \tag{13}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + DS(t) \tag{14}$$

where $\alpha, m_i, n_i, k_j, \tau_i (\tau_i \geq 0) \in \Re$; integers P, Q $(Q \leq P)$, $f(.)$ is the differentiable generic nonlinear function. x_{τ_i} and y_{τ_i} stand for delayed state variables $x(t-\tau_i)$ and $y(t-\tau_i)$, respectively. Note that, the form of $f(.)$ and the value of P are shared in both the master's and slave's equations. As shown in Eq. (13), the driving signal is constituted by sum of multiple nonlinear transforms of delayed state variable, and it is generated by driving signal generator (DSG) as illustrated in Fig. 7. The master's and slave's equations in Eqs. (12) and (14) with $\{P = 1,\ f(x) = \frac{x}{1+x^b}\}$ and $\{P = 1,\ f(x) = sin(x)\}$ turn out being the well-known Mackey-Glass (Mackey & Glass, 1977) and Ikeda systems (Ikeda & Matsumoto, 1987), respectively.

It is clear to observe from the proposed synchronization model given in Eqs. (12)-(14) that

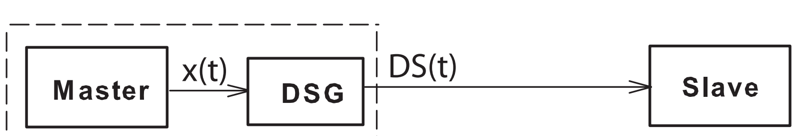


Fig. 7. The proposed synchronization model of MTDSs.

the structure of slave is identical to that of master, except for the presence of driving signal in the dynamical equation of slave.

In consideration to the synchronization condition, so far, there are two strategies (Pyragas, 1998a) which are used for dealing with synchronization of time-delay systems. The first one is based on the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) while the other one is based on the perturbation theory (Pyragas, 1998a). Note that, perturbation theory is used suitably for the case that the delay length is large ($\tau_i \to \infty$). And, the Krasovskii-Lyapunov theory can be used for the case of multiple time-delays as in this context. In the following subsections, this model is investigated with different synchronous schemes, i.e., lag, anticipating, projective-lag, and projective-anticipating.

3.1.1 Lag synchronization

As a general case, lag synchronization refers to the means that the slave's state variable is retarded with a time length in compared to the master's. Here, lag synchronization has been studied in coupled MTDSs described in Eqs. (12)-(14) with the desired synchronization manifold defined by

$$y(t) = x(t - \tau_d) \tag{15}$$

where $\tau_d \in \Re^+$ is a time-delay, called a manifold's delay. We define the synchronization error upon expected synchronization manifold in Eq. (15) as

$$\Delta(t) = y(t) - x(t - \tau_d) \tag{16}$$

And, the dynamics of synchronization error is

$$\frac{d\Delta}{dt} = \frac{dy}{dt} - \frac{dx(t-\tau_d)}{dt} \tag{17}$$

By applying the delay of τ_d to Eq. (12), we get $\frac{dx(t-\tau_d)}{dt} = -\alpha x(t-\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})$. Then, substituting $\frac{dx(t-\tau_d)}{dt}$, $y_{\tau_i} = x_{\tau_i+\tau_d} + \Delta_{\tau_i}$, and Eq. (14) into Eq. (17), the dynamics of synchronization error becomes

$$\begin{aligned}\frac{d\Delta}{dt} &= \frac{dy}{dt} - \frac{dx(t-\tau_d)}{dt} \\ &= \left[-\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] - \left[-\alpha x(t-\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})\right] \\ &= -\alpha\Delta + \sum_{i=1}^{P} n_i f(x_{\tau_i+\tau_d} + \Delta_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}}) - \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})\end{aligned} \tag{18}$$

$$\tau_{P+j} = \tau_i + \tau_d \tag{19}$$

It is assumed that delays in Eq. (18) are chosen so that Eq. (19) is satisfied. Hence, Eq. (18) is rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} n_i f(x_{\tau_i+\tau_d} + \Delta_{\tau_i}) - \sum_{i=1,j=1}^{P,Q} \left[m_i - k_j\right] f(x_{\tau_i+\tau_d}) \tag{20}$$

$$m_i - k_j = n_i \tag{21}$$

It is easy to realize that the derivative of $f(x+\delta) - f(x) = f'(x)\delta$ exists if $f(.)$ is differentiable, bounded, and δ is small enough. Also suppose that the value of coefficients in Eq. (20) is adopted so as the relation in Eq. (21) is fulfilled in pair. Note from Eqs. (19) and (21) that only some components in the master's and slave's equations are selected for such the relations. Therefore, Eq. (20) reduces to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} n_i f'(x_{\tau_i+\tau_d} + \Delta_{\tau_i})\Delta_{\tau_i} \tag{22}$$

By applying the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) to the case of multiple time-delays, the sufficient condition to achieve $\lim_{t\to\infty}\Delta(t) = 0$ from Eq. (22) is expressed as

$$\alpha > \sum_{i=1}^{P} |n_i| \sup \left| f'(x_{\tau_i+\tau_d}) \right| \tag{23}$$

where $\sup|f'(.)|$ stands for the supreme limit of $|f'(.)|$. It is easy to see that the sufficicent condition for synchronization is obtained under a series of assumptions. Noticably, the linear delayed system of Δ given in Eq. (22) is with time-dependent coefficients. The specific example shown in Section 4 with coupled modified Mackey-Glass systems will demonstrate and verify for the case.

Next, combination synchronous scheme will be presented, there, the mentioned synchronous scheme of coupled MTDSs is associated with projective one.

3.1.2 Projective-lag synchronization

In this section, the lag synchronization of coupled MTDSs is investigated in a way that the master's and slave's state variables correlate each other upon a scale factor. The dynamical equations for synchronous system are defined in Eqs. (12)- (14). The desired projective-lag manifold is described by

$$ay(t) = bx(t - \tau_d) \tag{24}$$

where a and b are nonzero real numbers, and τ_d is the time lag by which the state variable of the master is retarded in comparison with that of the slave. The synchronization error can be written as

$$\Delta(t) = ay(t) - bx(t - \tau_d), \tag{25}$$

And, dynamics of synchronization error is

$$\frac{d\Delta}{dt} = a\frac{dy}{dt} - b\frac{dx(t-\tau_d)}{dt}. \tag{26}$$

By substituting appropriate components to Eq. (26), the dynamics of synchronization error can be rewritten as

$$\frac{d\Delta}{dt} = a\left[-\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] - b\left[-\alpha x(t-\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})\right] \tag{27}$$

Moreover, y_{τ_i} can be deduced from Eq. (25) as

$$y_{\tau_i} = \frac{bx_{\tau_i+\tau_d} + \Delta_{\tau_i}}{a} \tag{28}$$

And, Eq. (27) can be represented as

$$\frac{d\Delta}{dt} = a\left[-\alpha y + \sum_{i=1}^{P} n_i f\left(\frac{bx_{\tau_i+\tau_d} + \Delta_{\tau_i}}{a}\right) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] - b\left[-\alpha x(t-\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})\right] \tag{29}$$

Let us assume that the relation of delays is as given in Eq. (19), $\tau_{P+j} = \tau_i + \tau_d$. The error dynamics in Eq. (29) becomes

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1,j=1}^{P,Q}\left[an_i f\left(\frac{bx_{\tau_i+\tau_d} + \Delta_{\tau_i}}{a}\right) - (bm_i - ak_j) f(x_{\tau_i+\tau_d})\right] \tag{30}$$

The right-hand side of Eq. (28) can be represented as

$$\frac{bx_{\tau_i+\tau_d} + \Delta_{\tau_i}}{a} = x_{\tau_i+\tau_d} + \Delta_{\tau_i^{(app)}} \tag{31}$$

where $\tau_i^{(app)}$ is a time-delay at which the synchronization error satisfies Eq. (31). By replacing right-hand side of Eq. (31) to Eq. (30), The error dynamics can be rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1,j=1}^{P,Q}\left[an_i f(x_{\tau_i+\tau_d} + \Delta_{\tau_i^{(app)}}) - (bm_i - ak_j) f(x_{\tau_i+\tau_d})\right] \tag{32}$$

Suppose that the relation of parameters in Eq. (32) as follows

$$bm_i - ak_j = an_i \tag{33}$$

If $\Delta_{\tau_i^{(app)}}$ is small enough and $f(.)$ is differentiable, bounded, then Eq. (32) can be reduced to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} an_i f'(x_{\tau_i+\tau_d})\Delta_{\tau_i^{(app)}} \tag{34}$$

By applying the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) to this case, the sufficient condition for synchronization is expressed as

$$\alpha > \sum_{i=1}^{P} |an_i| \sup\left|f'(x_{\tau_i+\tau_d})\right| \tag{35}$$

It is clear that the main difference of this scheme in comparison with lag synchronization is the existence of scale factor. This leads to the change in the synchronization condition. In fact, projective-lag synchronization becomes lag synchronization when scale factor is equivalent to unity, but the relative value of α is changed in the sufficient condition regarding to the bound. This allows us to arrange multiple slaves with the same structure which are synchronized with a certain master under various scale factors. Anyways, the value of n_i and k_j must be adjusted correspondingly. This can not be the case by using lag synchronization as presented in the previous section, that is, only one slave with a certain structure is satisfied.

3.1.3 Anticipating synchronization

In this section, anticipating synchronization of coupled MTDSs is presented, in which the master's motion can be anticipated by the slave's. The proposed model given in Eqs. (12)-(14) is investigated with the desired synchronization manifold of

$$y(t) = x(t+\tau_d) \tag{36}$$

where $\tau_d \in \Re^+$ is the time length of anticipation. It is also called a manifold's delay because the master's state variable is retarded in compared with the slave's. Synchronization error in this case is

$$\Delta(t) = y(t) - x(t+\tau_d) \tag{37}$$

Similar to the scheme of lag synchronization, the dynamics of synchronization error is written as

$$\frac{d\Delta}{dt} = \frac{dy}{dt} - \frac{dx(t+\tau_d)}{dt} \tag{38}$$

By substituting $\frac{dx(t+\tau_d)}{dt} = -\alpha x(t+\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i-\tau_d})$, $y_{\tau_i} = x_{\tau_i-\tau_d} + \Delta_{\tau_i}$, and $\frac{dy}{dt}$ into Eq. (38), the dynamics of synchronization error is described by

$$\begin{aligned}\frac{d\Delta}{dt} &= \frac{dy}{dt} - \frac{dx(t+\tau_d)}{dt}\\ &= \left[-\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] - \left[-\alpha x(t+\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i-\tau_d})\right]\\ &= -\alpha\Delta + \sum_{i=1}^{P} n_i f(x_{\tau_i-\tau_d} + \Delta_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}}) - \sum_{i=1}^{P} m_i f(x_{\tau_i-\tau_d})\end{aligned} \tag{39}$$

Assume that τ_{P+j} in Eq. (39) are fulfilled the relation of

$$\tau_{P+j} = \tau_i - \tau_d \tag{40}$$

delays must be non-negative, thus, τ_i must be equal to or greater than τ_d in Eq. (19). Equation (39) is represented as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} n_i f(x_{\tau_i-\tau_d} + \Delta_{\tau_i}) - \sum_{i=1,j=1}^{P,Q} \left[m_i - k_j\right] f(x_{\tau_i-\tau_d}) \tag{41}$$

Applying the same reasoning in lag synchronization to this case, parameters satisfies the relation given in Eq. (21). Equation (41) reduces to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} n_i f'(x_{\tau_i-\tau_d})\Delta_{\tau_i} \tag{42}$$

And, the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) is applied to Eq. (42), hence, the sufficient condition for synchronization for anticipating synchronization is

$$\alpha > \sum_{i=1}^{P} |n_i| \,\mathsf{sup}\left|f'(x_{\tau_i-\tau_d})\right| \tag{43}$$

It is clear from (35) and (43) that there is small difference made to the relation of delays in comparison to lag synchronization, and a completely new scheme is resulted. Therefore, the switching between schemes of lag and anticipating synchronization can be obtained in such a simple way. This may be exploited for various purposes including secure communications.

3.1.4 Projective-anticipating synchronization

Obviously, projective-anticipating synchronization is examined in a very similar way to that dealing with the scheme of projective-lag synchronization. The dynamical equations for synchronous system are as given in Eq. (12)- (14). The considered projective-anticipating manifold is as

$$ay(t) = bx(t + \tau_d) \tag{44}$$

where a and b are nonzero real numbers, and τ_d is the time lag by which the state variable of the slave is retarded in comparison with that of the master. The synchronization error is defined as

$$\Delta = ay - bx(t + \tau_d) \tag{45}$$

Dynamics of synchronization error is as

$$\frac{d\Delta}{dt} = a\frac{dy}{dt} - b\frac{dx(t+\tau_d)}{dt}. \tag{46}$$

By substituting $\frac{dy}{dt}$ and $\frac{dx(t+\tau_d)}{dt}$ to Eq. (46), the dynamics of synchronization error becomes

$$\frac{d\Delta}{dt} = a\left[-\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] - b\left[-\alpha x(t+\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i - \tau_d})\right] \tag{47}$$

It is clear that y_{τ_i} can be deduced from Eq. (45) as

$$y_{\tau_i} = \frac{bx_{\tau_i - \tau_d} + \Delta_{\tau_i}}{a} \tag{48}$$

Hence, Eq. (47) can be represented as

$$\begin{aligned}\frac{d\Delta}{dt} &= a\left[-\alpha y + \sum_{i=1}^{P} n_i f\left(\frac{bx_{\tau_i - \tau_d} + \Delta_{\tau_i}}{a}\right) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] \\ &\quad - b\left[-\alpha x_{\tau_d} + \sum_{i=1}^{P} m_i f(x_{\tau_i - \tau_d})\right]\end{aligned} \tag{49}$$

Similar to anticipating synchronization, the relation of delays is chosen as given in Eq. (40), $\tau_{P+j} = \tau_i - \tau_d$. The error dynamics in Eq. (49) is rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1,j=1}^{P,Q}\left[an_i f\left(\frac{bx_{\tau_i - \tau_d} + \Delta_{\tau_i}}{a}\right) - (bm_i - ak_j) f(x_{\tau_i - \tau_d})\right] \tag{50}$$

The right-hand side of Eq. (48) can be equivalent to

$$\frac{bx_{\tau_i - \tau_d} + \Delta_{\tau_i}}{a} = x_{\tau_i - \tau_d} + \Delta_{\tau_i^{(app)}} \tag{51}$$

where $\tau_i^{(app)}$ is a time-delay satisfying Eq. (51). Therefore, the error dynamics can be rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1,j=1}^{P,Q} \left[an_i f(x_{\tau_i - \tau_d} + \Delta_{\tau_i^{(app)}}) - (bm_i - ak_j) f(x_{\tau_i - \tau_d}) \right] \tag{52}$$

Suppose that the relation of parameters in Eq. (52) is as given in Eq. (33), $bm_i - ak_j = an_i$. $\Delta_{\tau_i^{(app)}}$ is small enough, $f(.)$ is differentiable and bounded, hence, Eq. (52) is reduced to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} an_i f'(x_{\tau_i - \tau_d}) \Delta_{\tau_i^{(app)}} \tag{53}$$

The sufficient condition for synchronization can be expressed as

$$\alpha > \sum_{i=1}^{P} |an_i| \sup \left| f'(x_{\tau_i - \tau_d}) \right| \tag{54}$$

It is easy to see that the change from anticipating into projective-anticipating synchronization is similar to that from lag to projective-lag one. It is realized that transition from the lag to anticipating is simply done by changing the relation of delays. This is easy to be observed on their sufficient conditions.

3.2 Synchronization of coupled nonidentical MTDSs

It is easy to observe from the synchronization model presented in Eqs. (12)-(14) that the value of P and the function form of $f(.)$ are shared in the master's and slave's equations. It means that the structure of the master is identical to that of slave. In other words, the proposed synchronization model above is not a truly general one. In this section, the proposed synchronization model of coupled nonidentical MTDSs is presented, there, the similarity in the master's and slave's equations is removed. The dynamical equations representing for the synchronization are defined as

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}}) \tag{55}$$

Driving signal:

$$DS(t) = \sum_{j=1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) \tag{56}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + DS(t) \tag{57}$$

where $\alpha, m_i, n_i, k_j, \tau_i^{(M)}, \tau_j^{(DS)}, \tau_i^{(S)} \in \Re$; P, Q and R are integers. The delayed state variables $x_{\tau_i^{(M)}}$, $x_{\tau_j^{(DS)}}$ and $y_{\tau_i^{(S)}}$ stand for $x(t - \tau_i^{(M)})$, $x(t - \tau_j^{(DS)})$ and $y(t - \tau_i^{(S)})$, respectively. $f_i^{(M)}(.)$, $f_j^{(DS)}(.)$ and $f_i^{(S)}(.)$ are differentiable, generic, and nonlinear functions. The superscripts (M), (S) and (DS) associated with main symbols (delay, function, set of function forms) indicate that they are belonged to the master, slave and driving signal, respectively.

The non-identicalness between the master's and slave's configuration can be clarified by defining the set of function forms, $S = \{F_i; i = 1..N\}$, in which F_i ($i = 1..N$) represents for the function form of $f_i^{(M)}(.)$, $f_j^{(DS)}(.)$ and $f_i^{(S)}(.)$ in Eqs. (55)-(57). The subsets of S_M, S_S and S_{DSG} are collections of function forms of the master, slave and DSG, respectively. It is assumed that the relation among subsets is $S_{DSG} \subseteq S_M \cup S_S$. It is easy to realize that the structure of master is completely nonidentical to that of slave if $S_I = S_M \cap S_S \equiv \Phi$. Otherwise, if there are I components of nonlinear transforms whose function forms and value of delays are shared between the master's and slave's equations, i.e., $S_I = S_M \cap S_S \neq \Phi$ and $\tau_i^{(M)} = \tau_i^{(S)}$ for $i = 1..I$. These components are called *identicalness* ones which make pairs of $\{f^{(M)}(x_{\tau_i^{(M)}})$ *vs.* $f^{(S)}(y_{\tau_i^{(S)}})\}$ for $i = 1..I$.

Therefore, there are two cases needed to consider specifically: *(i)* the structure of master is partially identical to that of slave by means of identicalness components, and *(ii)* the structure of master is completely nonidentical to that of slave. In any cases, it is easy to realize from the relation among S_M, S_S and S_{DSG} that the difference between the master's and slave's equations can be complemented by the DSG's equation. In other words, function forms and value of parameters will be chosen appropriately for the driving signal's equation so that the Krasovskii-Lyapunov theory can be used for considering the synchronization condition in a certain case. For simplicity, only scheme of lag synchronization with the synchronization manifold of $y(t) = x(t - \tau_d)$ is studied, and other schemes can be extended as in a way of synchronization of coupled identical MTDSs.

3.2.1 Structure of master partially identical to that of slave

Suppose that there are I identicalness components shared between the master's and slave's equations, hence, Eqs. (55) and (57) can be decomposed as

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{I} m_i f_i^{(M)}(x_{\tau_i^{(M)}}) + \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}}) \tag{58}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{I} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + DS(t) \tag{59}$$

where $f_i^{(M)}$ is with the form identical to $f_i^{(S)}$ and $\tau_i^{(M)} = \tau_i^{(S)}$ for $i = 1..I$. They are pairs of identicalness components. The driving signal's equation in Eq. (56) is chosen in the following form

$$DS(t) = \sum_{j=1}^{I} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) + \sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) \tag{60}$$

where forms of $f_j^{(DS)}(.)$ for $j = 1..I$ are, in pair, identical to that of $f_i^{(M)}$ as well as of $f_i^{(S)}$ for $i = 1..I$. Let's consider the lag synchronization manifold of

$$y(t) = x(t - \tau_d) \tag{61}$$

And, the synchronization error is

$$\Delta(t) = y(t) - x(t - \tau_d) \tag{62}$$

Hence, the dynamics of synchronization error is expressed by

$$\begin{aligned}\frac{d\Delta}{dt} &= \frac{dy}{dt} - \frac{dx(t-\tau_d)}{dt} \\ &= -\alpha y + \sum_{i=1}^{I} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{j=1}^{I} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) + \\ &+ \sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) + \alpha x(t-\tau_d) - \sum_{i=1}^{I} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) - \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d})\end{aligned} \tag{63}$$

By applying delay of $\tau_i^{(S)}$ to Eq. (62), $y_{\tau_i^{(S)}}$ can be deduced as

$$y_{\tau_i^{(S)}} = x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}} \tag{64}$$

By substituting $y_{\tau_i}^{(S)}$ to Eq. (63), the dynamics of synchronization error can be rewritten as

$$\begin{aligned}\frac{d\Delta}{dt} &= -\alpha\Delta + \sum_{i=1}^{I} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) + \sum_{j=1}^{I} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) \\ &+ \sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) - \sum_{i=1}^{I} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) - \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d})\end{aligned} \tag{65}$$

Suppose that the relation of delays in the fourth and sixth terms at the right-hand side of Eq. (65) is

$$\tau_j^{(DS)} = \tau_i^{(M)} + \tau_d \quad (\equiv \tau_i^{(S)} + \tau_d) \quad for\ j,i = 1..I \tag{66}$$

Hence, Eq. (65) can be reduced to

$$\begin{aligned}\frac{d\Delta}{dt} &= -\alpha\Delta + \sum_{i=1}^{I} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) - \sum_{i=1}^{I}(m_i - k_i) f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) + \sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) - \\ &- \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}})\end{aligned} \tag{67}$$

Also suppose that function forms and value of parameters of the fourth term of Eq. (67) (the second right-hand term of Eq. (60)) are chosen so that the last three terms of Eq. (67) satisfy the following equation

$$\sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) - \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) = 0 \tag{68}$$

Let us assume that $Q = P + R - I$. The first left-hand term is decomposed, and Eq. (68) becomes

$$\begin{aligned}&\sum_{j1=1}^{P-I} k_{I+j1} f_{I+j1}^{(DS)}(x_{\tau_{I+j1}^{(DS)}}) + \sum_{j2=1}^{R-I} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}) \\ &- \sum_{i=1}^{P-I} m_{I+i} f_{I+i}^{(M)}(x_{\tau_{I+i}^{(M)}+\tau_d}) + \sum_{i=1}^{R-I} n_{I+i} f_{I+i}^{(S)}(x_{\tau_{I+i}^{(S)}+\tau_d} + \Delta_{\tau_{I+i}^{(S)}}) = 0\end{aligned} \tag{69}$$

Undoubtedly, Eq. (69) can be fulfilled if following assumptions are made: $k_{I+j1} = m_{I+i}$, $\tau_{I+j1}^{(DS)} = \tau_{I+i}^{(M)} + \tau_d$ and forms of $f_{I+j1}^{(DS)}(.)$ are identical to that of $f_{I+i}^{(M)}(.)$ for $i, j1 = 1..(P - I)$, and $k_{P+j2} = -n_{I+i}$, $\tau_{P+j2}^{(DS)} = \tau_{I+i}^{(S)} + \tau_d$, $\Delta_{\tau_{I+i}^{(S)}}$ is equal to zero as well as the form of $f_{P+j2}^{(DS)}(.)$ is identical to that of $f_{I+i}^{(S)}(.)$ for $i, j2 = 1..(R - I)$. Thus, Eq. (67) can be represented as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{I} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) - \sum_{i=1}^{I}(m_i - k_i) f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) \tag{70}$$

According to above assumptions, $\tau_i^{(S)} = \tau_i^{(M)}$ and forms of $f_i^{(M)}(.)$ being identical to those of $f_i^{(M)}(.)$ for $i = 1..I$ have been made. Here, further suppose that functions $f_i^{(M)}(.)$ and $f_i^{(S)}(.)$ are bounded. If synchronization errors $\Delta_{\tau_i^{(S)}}$ are small enough and $m_i - k_j = n_i$ for $i = 1..I$, Eq. (70) can be reduced to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{I} n_i f_i^{(S)'}(x_{\tau_i^{(S)}+\tau_d})\Delta_{\tau_i^{(S)}} \tag{71}$$

where $f_i^{(S)'}(.)$ is the derivative of $f_i^{(S)}(.)$. By applying the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) to the case of multiple time-delays in Eq. (71), the sufficient condition for synchronization can be expressed as

$$\alpha > \sum_{i=1}^{I} |n_i| \sup \left| f_i^{(S)'}(x_{\tau_i^{(S)}+\tau_d}) \right| \tag{72}$$

It turns out that the difference in the structures of the master and slave can be complemented in the equation of driving signal. In order to test the proposed scheme, Example 5 is demonstrated in Section 4, in which the master's equation is in the heterogeneous form and the slave's is in the multiple time-delay Ikeda equation.

3.2.2 Structure of master completely nonidentical to that of slave

In this section, the synchronous system given in Eqs. (58)-(59) is examined, in which there is no identicalness component shared between the master's and slave's equations. In other words, the function set is of $S_I = S_M \cap S_S = \Phi$. Therefore, the driving signal's equation must contain all function forms of the master's and slave's equations or $S_{DSG} = S_M \cup S_S$ and $Q = P + R$. The driving signal's equation Eq. (56) can be decomposed to

$$DS(t) = \sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}}) + \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}) \tag{73}$$

And, the synchronization error Eq. (62) can be represented as below

$$\begin{aligned}\frac{d\Delta}{dt} &= \frac{dy}{dt} - \frac{dx(t-\tau_d)}{dt}\\ &= -\alpha y + \sum_{i=1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}})\\ &+ \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}) + \alpha x(t-\tau_d) - \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d})\\ &= -\alpha\Delta + \sum_{i=1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}\\ &+ \sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}}) - \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d})\end{aligned} \tag{74}$$

By substituting $y_{\tau_s^{(S)}}$ from Eq. (64) into Eq. (74), the dynamics of synchronization error is rewritten as

$$\begin{aligned}\frac{d\Delta}{dt} &= -\alpha\Delta + \sum_{i=1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) + \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}})\\ &+ \sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}}) - \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d})\end{aligned} \tag{75}$$

Assume that value of parameters and function forms of the first right-hand term of Eq. (73) are chosen so that the relation between the last two right-hand terms of Eq. (75) is as

$$\sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}}) - \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) = 0 \tag{76}$$

Equation Eq. (76) is fulfilled if the relation is as $k_{j1} = m_i$, $\tau_{j1}^{(DS)} = \tau_i^{(M)} + \tau_d$ and the form of $f_{j1}^{(DS)}(.)$ is identical to that of $f_i^{(M)}(.)$ for $i, j1 = 1..P$. At this point, the dynamics of synchronization error in (75) can be reduced to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) + \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}) \tag{77}$$

As mentioned, the form of $f_i^{(S)}(.)$ is identical to that of $f_{P+j2}^{(DS)}(.)$ in pair. Here, we suppose that coefficients and delays in Eq. (77) are adopted as $k_{P+j2} = -n_i$ and $\tau_{P+j2}^{(DS)} = \tau_i^{(S)} + \tau_d$ for $i, j2 = 1..P$. If $\Delta_{\tau_i^{(S)}}$ is small enough and functions $f_i^{(S)}$ are bounded, Eq. (77) can be rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{R} n_i f_i^{(S)'}(x_{\tau_i^{(S)}+\tau_d})\Delta_{\tau_i^{(S)}} \tag{78}$$

where $f_i^{(S)'}(.)$ is the derivative of $f_i^{(S)}(.)$. Similarly, the synchronization condition is obtained by applying the Krasovskii-Lyapunov (Hale & Lunel, 1993; Krasovskii, 1963) theory to Eq. (78); that is

$$\alpha > \sum_{i=1}^{R} |n_i| \sup \left| f_i^{(S)'}(x_{\tau_i^{(S)}+\tau_d}) \right| \tag{79}$$

It is undoubtedly that for a certain master and slave in the form of MTDS, we always obtained synchronous regime. Example 6 in Section 4 is given to verify for synchronization of completely nonidentical MTDSs; the multidelay Mackey-Glass and multidelay Ikeda systems.

4. Numerical simulation for synchronous schemes on the proposed models

In this subsection, a number of specific examples demonstrate and verify for the general description. Each example will correspond to a proposal in above section.

Example 1:
This example illustrates the lag synchronous scheme in coupled identical MTDSs given in Section 3.1.1. Let's consider the synchronization of coupled six-delays Mackey-Glass systems with the coupling signal constituted by the four-delays components. The dynamical equations are as
Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P=6} m_i \frac{x_{\tau_i}}{1+x_{\tau_i}^b} \tag{80}$$

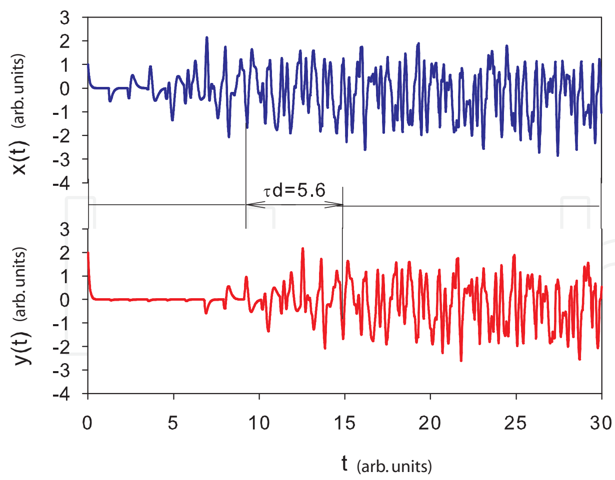
Driving signal:

$$DS(t) = \sum_{j=1}^{Q=4} k_j \frac{x_{P+j}}{1+x_{\tau_{P+j}}^b} \tag{81}$$

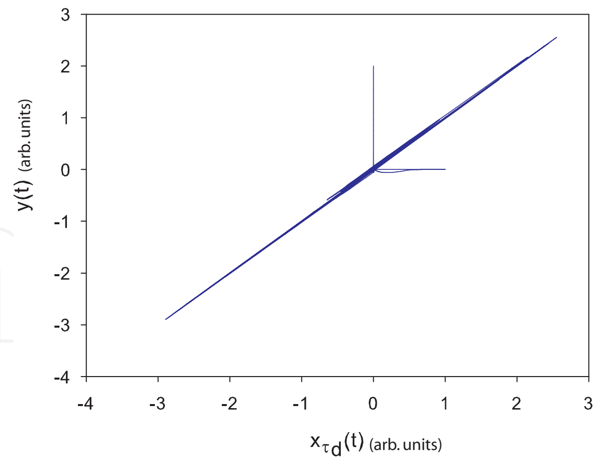
Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P=6} n_i \frac{x_{\tau_i}}{1+x_{\tau_i}^b} + DS(t) \tag{82}$$

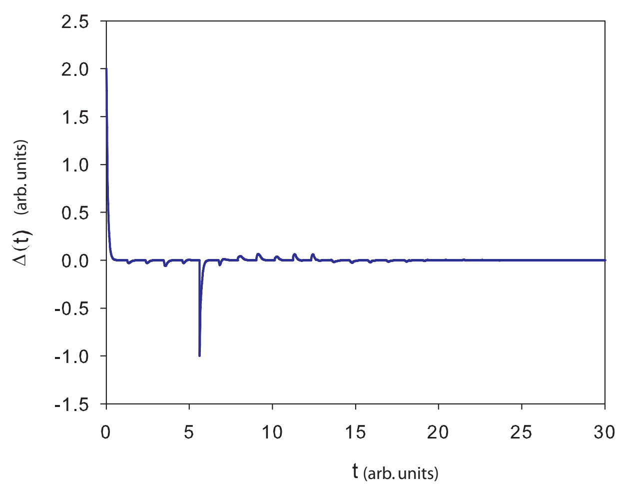
Moreover, the supreme limit of the function $f'(x)$ is equal to $\frac{(b-1)^2}{4b}$ at $x = \left(\frac{b+1}{b-1}\right)^{\frac{1}{b}}$ (Pyragas, 1998a). The relation of delays and of parameters is chosen as: $\tau_7 = \tau_1 + \tau_d$, $\tau_8 = \tau_2 + \tau_d$, $\tau_9 = \tau_4 + \tau_d$, $\tau_{10} = \tau_5 + \tau_d$, $m_1 - k_1 = n_1$, $m_2 - k_2 = n_2$, $m_3 = n_3$, $m_4 - k_3 = n_4$, $m_5 - k_4 = n_5$, $m_6 = n_6$.
The value of delays and parameters are adopted as: $b = 10$, $\alpha = 12.3$, $m_1 = -20.0$, $m_2 = -15.0$, $m_3 = -1.0$, $m_4 = -16.0$, $m_5 = -25.0$, $m_6 = -1.0$, $n_1 = -1.0$, $n_2 = -1.0$, $n_3 = -1.0$, $n_4 = -1.0$, $n_5 = -1.0$, $n_6 = -1.0$, $k_1 = -19.0$, $k_2 = -14.0$, $k_3 = -15.0$, $k_4 = -24.0$, $\tau_d = 5.6$, $\tau_1 = 1.2$, $\tau_2 = 2.3$, $\tau_3 = 3.4$, $\tau_4 = 4.5$, $\tau_5 = 5.6$, $\tau_6 = 6.7$, $\tau_7 = 6.8$, $\tau_8 = 7.9$, $\tau_9 = 10.1$, $\tau_{10} = 11.2$. Illustrated in Fig. 8 is the simulation result for the synchronization manifold of $y(t) = x(t-5.6)$. Obviously, the lag existing in the state variables is observed in Fig. 8(a). Establishment of the synchronization manifold can be seen through the portrait of $x(t-5.6)$ versus $y(t)$ in Fig. 8(b). Moreover, the synchronization error vanishes in time evolution as shown in Fig. 8(c). As a result, the desired synchronization manifold is firmly achieved.



(a) Time series of $x(t)$ and $y(t)$



(b) Portrait of $x(t-5.6)$ versus $y(t)$



(c) Synchronization error $\Delta(t) = y(t) - x(t-5.6)$

Fig. 8. Simulation result of lag synchronization of coupled six-delays Mackey-Glass systems.

Example 2:

This example demonstrates the description of anticipating synchronization of coupled identical MTDSs given in Section 3.1.3. The anticipating synchronous scheme is examined in coupled four-delays Ikeda systems with the dynamical equations given as follows

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P=4} m_i \sin x_{\tau_i} \tag{83}$$

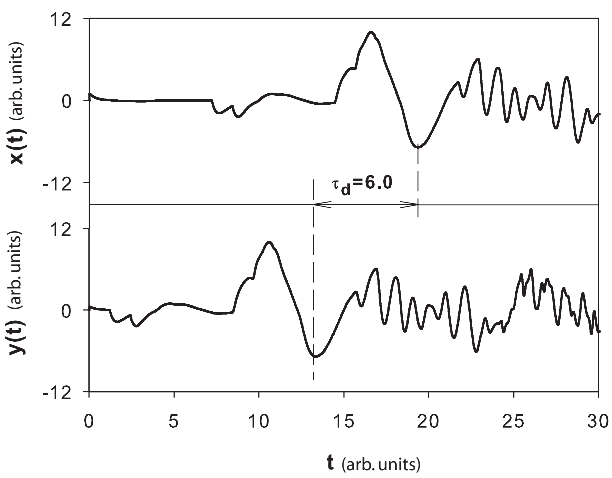
Driving signal:

$$DS(t) = \sum_{j=1}^{Q=2} k_j \sin x_{\tau_{P+j}} \tag{84}$$

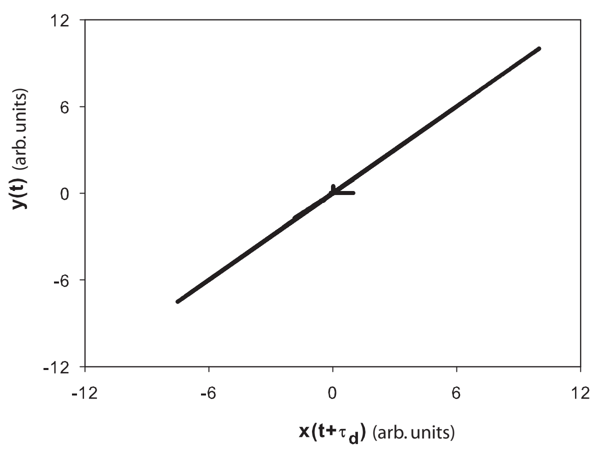
Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P=4} n_i \sin y_{\tau_i} + DS(t) \tag{85}$$

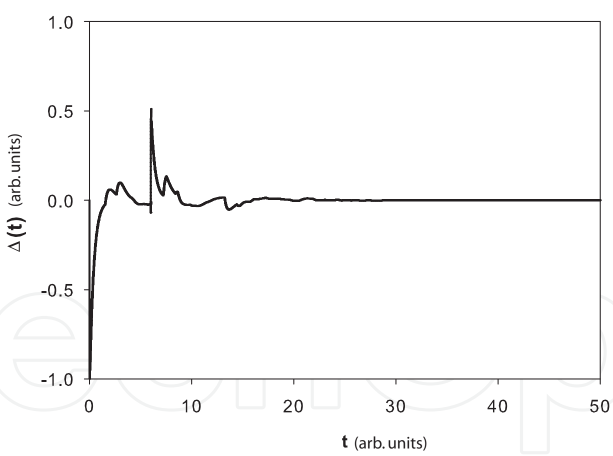
Following to above description, the relation of parameters and delays is chosen as: $m_1 = n_1$, $m_2 - k_1 = n_2$, $m_3 = n_3$, $m_4 - k_2 = n_4$, $\tau_5 = \tau_2 - \tau_d$, $\tau_6 = \tau_4 - \tau_d$. Anticipating synchronization manifold considered in this example is $y(t) = x(t + \tau_d)$, and chosen $\tau_d = 6.0$. The adopted value of parameters and delays for simulation are as: $\alpha = 2.5$, $m_1 = -0.5$, $m_2 = -13.5$, $m_3 = -0.6$, $m_4 = -14.6$, $n_1 = -0.5$, $n_2 = -0.9$, $n_3 = -0.6$, $n_4 = -0.2$, $k_1 = -12.6$, $k_2 = -14.4$, $\tau_1 = 1.5$, $\tau_2 = 7.2$, $\tau_3 = 2.6$, $\tau_4 = 8.4$, $\tau_5 = 1.2$, $\tau_6 = 2.4$.
The simulation result is displayed in Fig. 9. It is realized from Fig. 9(a) that the slave anticipates the master's motion, and the synchronization manifold of $y(t) = x(t + 6.0)$ is established as illustrated in Fig. 9(b), with vanishing synchronization error as depicted in Fig. 9(c).



(a) Time series of $x(t)$ and $y(t)$



(b) Portrait of $x(t + 6.0)$ versus $y(t)$



(c) Synchronization error $\Delta(t) = y(t) - x(t + 6.0)$

Fig. 9. Simulation result of anticipating synchronization of coupled four-delays Ikeda systems

Example 3:
To support for projective-lag synchronization as given in Section 3.1.2, this example deals with synchronization of coupled six-delays Mackey-Glass systems with the driving signal constituted by the four-delays components. The dynamical equations are expressed in Eqs. (80)- (82). For the synchronization manifold of $ay(t) = bx(t - \tau_d)$, the relations between

the value of delays and parameters are chosen as $\tau_7 = \tau_d + \tau_1$, $\tau_8 = \tau_d + \tau_2$, $\tau_9 = \tau_d + \tau_4$, $\tau_{10} = \tau_d + \tau_6$, $bm_1 - ak_1 = an_1$, $bm_2 - ak_2 = an_2$, $m_3 = n_3$, $bm_4 - ak_3 = an_4$, $m_5 = n_5$, $bm_6 - ak_4 = an_6$. According to Eq. (35), the sufficient condition for synchronization is

$$\alpha > \sum_{i=1}^{P=6} |an_i| \sup \left| f'(x_{\tau_i + \tau_d}) \right| . \tag{86}$$

The value of delays and parameters adopted for simulation are $a = 1.0$, $b = 3.0$, $c = 10$, $\alpha = 6.3$, $\tau_d = 5.6$, $\tau_1 = 6.7$, $\tau_2 = 3.4$, $\tau_3 = 4.5$, $\tau_4 = 5.6$, $\tau_5 = 2.3$, $\tau_6 = 1.2$, $\tau_7 = 12.3$, $\tau_8 = 9.0$, $\tau_9 = 11.2$, $\tau_{10} = 6.8$, $m_1 = -8.0$, $m_2 = -7.0$, $m_3 = -0.3$, $m_4 = -6.7$, $m_5 = -0.2$, $m_6 = -5.4$, $n_1 = -0.6$, $n_2 = -0.5$, $n_3 = -0.3$, $n_4 = -0.8$, $n_5 = -0.2$, $n_6 = -0.7$, $k_1 = -23.4$, $k_2 = -20.5$, $k_3 = -19.3$, and $k_4 = -15.5$.
The simulation result is illustrated in Fig. 10 with synchronization manifold of $1.0y(t) = 3.0x(t - 5.6)$. The scale factor can be seen by means of the scale of vertical axes in Fig. 10(a). The scale factor can also be observed via the slope of the synchronization line in the portrait of $x(t - 5.6)$ versus $y(t)$ shown in Fig. 10(b). Moreover, the synchronization error is reduced with respect to time as displayed in Figs. 10(c). However, the level of $\Delta_{\tau_i}^{(app)}$ in the linear approximation given in Eq. (31) is dependent on the difference between the value of a and b, $\delta = a - b$. Therefore, examination on the impact of $\delta = a - b$ on the synchronization error is necessary. As presented in Fig. 10(d) is the relation between the means square error (MSE) of the synchronization error in whole synchronizing time and $\delta = a - b$. It is clear that synchronization error is lowest when $\delta = 0$ or $a = b$.

Example 4:
The description given in Section 3.1.4 is illustrated in this example. Projective-anticipating synchronization of coupled five-delays Mackey-Glass systems is examined with three-delays driving signal. The dynamical equations are as
Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P=5} m_i \frac{x_{\tau_i}}{1 + x_{\tau_i}^c} \tag{87}$$

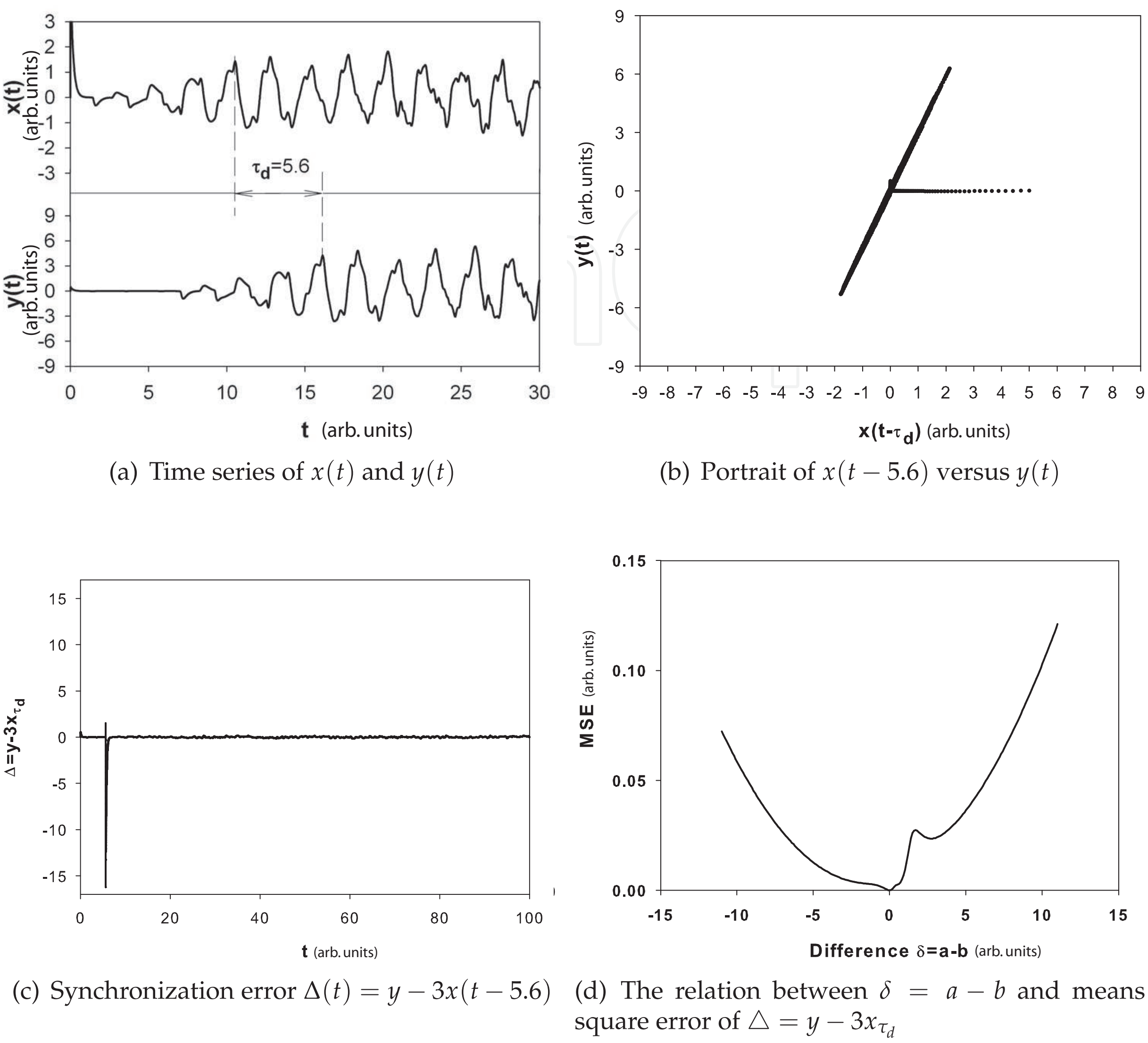
Driving signal:

$$DS(t) = \sum_{j=1}^{Q=3} k_j \frac{x_{\tau_{P+j}}}{1 + x_{\tau_{P+j}}^c} \tag{88}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P=5} n_i \frac{y_{\tau_i}}{1 + y_{\tau_i}^c} + DS(t) \tag{89}$$

The synchronization manifold of $ay(t) = bx(t + \tau_d)$ is studied with the relation of delays and parameters chosen as: $\tau_6 = \tau_1 - \tau_d$, $\tau_7 = \tau_3 - \tau_d$, $\tau_8 = \tau_5 - \tau_d$, $bm_1 - ak_1 = an_1$, $m_2 = n_2$, $bm_3 - ak_2 = an_3$, $m_4 = n_4$, $bm_5 - ak_3 = an_5$. The value of parameters and delays for simulation is set at: $a = -2.5$, $b = 1.5$, $\alpha = 16.3$, $c = 10$, $m_1 = -16.2$, $m_2 = -0.3$, $m_3 = -14.5$, $m_4 = -1.0$, $m_5 = -18.6$, $n_1 = -0.4$, $n_2 = -0.3$, $n_3 = -0.8$, $n_4 = -1.0$, $n_5 = -0.7$, $k_1 = 10.12$, $k_2 = 9.5$, $k_3 = 11.86$, $\tau_d = 4.6$, $\tau_1 = 4.8$, $\tau_2 = 3.8$, $\tau_3 = 6.2$, $\tau_4 = 5.5$, $\tau_5 = 4.6$, $\tau_6 = 0.6$, $\tau_7 = 2.0$, $\tau_8 = 0.4$.
The simulation result is depicted in Fig. 11 with the synchronization manifold of $-2.5y(t) = 1.5x(t + 4.6)$. It is easy to observed the scale factor by means of the scale of vertical axes in



(a) Time series of $x(t)$ and $y(t)$

(b) Portrait of $x(t-5.6)$ versus $y(t)$

(c) Synchronization error $\Delta(t) = y - 3x(t-5.6)$

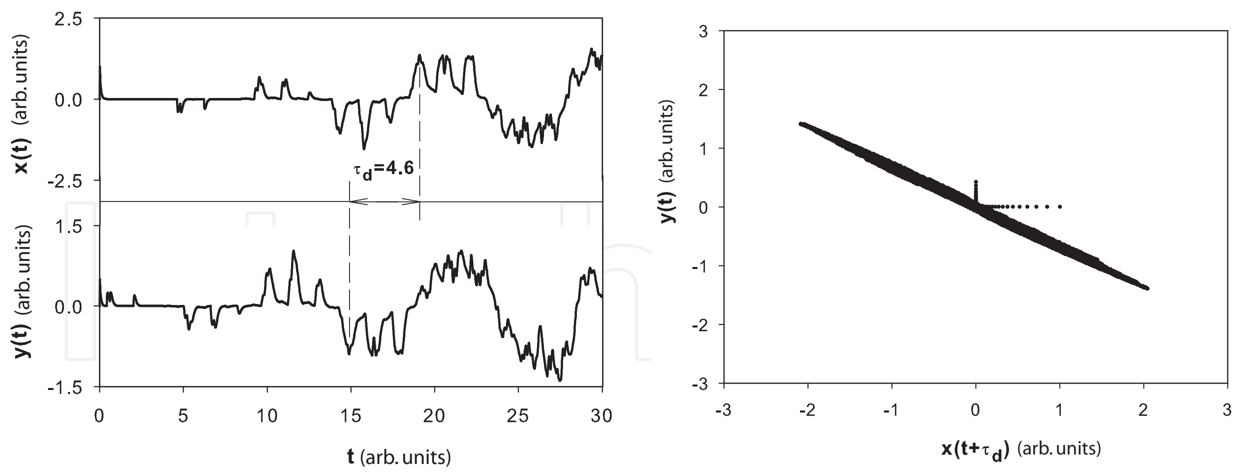
(d) The relation between $\delta = a - b$ and means square error of $\triangle = y - 3x_{\tau_d}$

Fig. 10. Simulation result of projective-lag synchronization of coupled six-delays Mackey-Glass systems

Fig. 11(a). The scale factor can also be observed via the slope of the line illustrated in the portrait of $x(t+4.6)$ versus $y(t)$ in Fig. 11(b).

Example 5:
Synchronization model in this example demonstrate the lag synchronization of partially identical MTDSs with the general description has been presented in Section 3.2.1. The master's and slave's equations are chosen as
Master:

$$\begin{aligned}\frac{dx}{dt} &= -\alpha x + m_1 sinx_{\tau_1^{(M)}} + m_2 sinx_{\tau_2^{(M)}} + m_3 sinx_{\tau_3^{(M)}} + \\ &+ m_4 \frac{x_{\tau_4^{(M)}}}{1 + x^8_{\tau_4^{(M)}}} + m_5 \frac{x_{\tau_5^{(M)}}}{1 + x^{10}_{\tau_5^{(M)}}}\end{aligned} \tag{90}$$



(a) Time series of $x(t)$ and $y(t)$

(b) Portrait of $x(t+4.6)$ versus $y(t)$

Fig. 11. Simulation result of projective-anticipating synchronization of coupled five-delays Mackey-Glass systems

Slave:

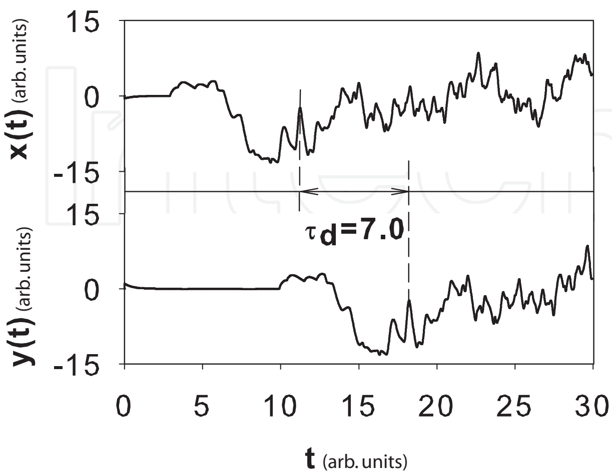
$$\begin{aligned} \frac{dy}{dt} &= -\alpha y + n_1 siny_{\tau_1^{(S)}} + n_2 siny_{\tau_2^{(S)}} + \\ &+ n_3 siny_{\tau_3^{(S)}} + n_4 siny_{\tau_4^{(S)}} + DS(t) \end{aligned} \tag{91}$$

It is easy to observe that the sets of function forms are $S_M = \{sinz, \frac{z}{1+z^8}, \frac{z}{1+z^{10}}\}$, $S_S = \{sinz\}$. Thus, $S_I = S_M \cap S_S = \{sinz\}$ and $S_{DSG} \subseteq S_M \cup S_S = \{sinz, \frac{z}{1+z^8}, \frac{z}{1+z^{10}}\}$. It is assumed that $\tau_1^{(M)} = \tau_1^{(S)}$ and $\tau_2^{(M)} = \tau_2^{(S)}$, thus, the pairs of identicalness components are $\{sinx_{\tau_1^{(M)}}\ vs.\ siny_{\tau_1^{(S)}}\}$ and $\{sinx_{\tau_2^{(M)}}\ vs.\ siny_{\tau_2^{(S)}}\}$. Therefore, the equation for driving signal must be chosen as

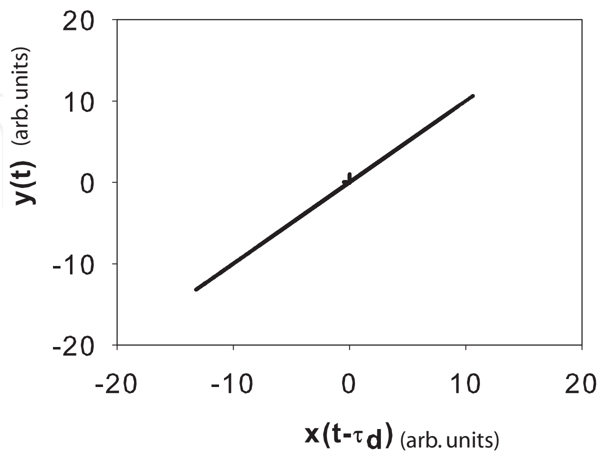
$$\begin{aligned} DS(t) &= k_1 sinx_{\tau_1^{(DS)}} + k_2 sinx_{\tau_2^{(DS)}} + k_3 sinx_{\tau_3^{(DS)}} + \\ &+ k_4 \frac{x_{\tau_4^{(DS)}}}{1 + x^8_{\tau_4^{(DS)}}} + k_5 \frac{x_{\tau_5^{(DS)}}}{1 + x^{10}_{\tau_5^{(DS)}}} + k_6 sinx_{\tau_6^{(DS)}} + k_7 sinx_{\tau_7^{(DS)}} \end{aligned} \tag{92}$$

Following to the assumption described in the above description for the manifold of $y(t) = x(t - \tau_d)$, the relation of delays and coefficients is chosen as: $m_1 - k_1 = n_1, m_2 - k_2 = n_2, k_3 = m_3, k_4 = m_4, k_5 = m_5, k_6 = -n_3, k_7 = -n_4$, $\tau_1^{(DS)} = \tau_1^{(M)} + \tau_d$ $(= \tau_1^{(S)} + \tau_d)$, $\tau_2^{(DS)} = \tau_2^{(M)} + \tau_d$ $(= \tau_2^{(S)} + \tau_d)$, $\tau_3^{(DS)} = \tau_3^{(M)} + \tau_d$, $\tau_4^{(DS)} = \tau_4^{(M)} + \tau_d$, $\tau_5^{(DS)} = \tau_5^{(M)} + \tau_d$, $\tau_6^{(DS)} = \tau_3^{(S)} + \tau_d$, and $\tau_7^{(DS)} = \tau_4^{(S)} + \tau_d$. In simulation, the value of parameters are adopted as: $\alpha = 2.0$, $m_1 = -15.4$, $m_2 = -16.0$, $m_3 = -0.35$, $m_4 = -20.0$, $m_5 = -18.5$, $n_1 = -0.2$, $n_2 = -0.1$, $n_3 = -0.25$, $n_4 = -0.4$, $k_1 = -15.2$, $k_2 = -15.9$, $k_3 = -0.35$, $k_4 = -20.0$, $k_5 = -18.5$, $k_6 = 0.25$, $k_7 = 0.4$, $\tau_1^{(M)} = 3.4$, $\tau_2^{(M)} = 4.5$, $\tau_3^{(M)} = 6.5$, $\tau_4^{(M)} = 5.3$, $\tau_5^{(M)} = 2.9$, $\tau_1^{(S)} = 3.4$, $\tau_2^{(S)} = 4.5$, $\tau_3^{(S)} = 2.0$, $\tau_4^{(S)} = 7.3$, $\tau_1^{(DS)} = 10.4$, $\tau_2^{(DS)} = 11.5$, $\tau_3^{(DS)} = 13.5$, $\tau_4^{(DS)} = 12.3$, $\tau_5^{(DS)} = 9.9$, $\tau_6^{(DS)} = 9.0$, and $\tau_7^{(DS)} = 14.3$.

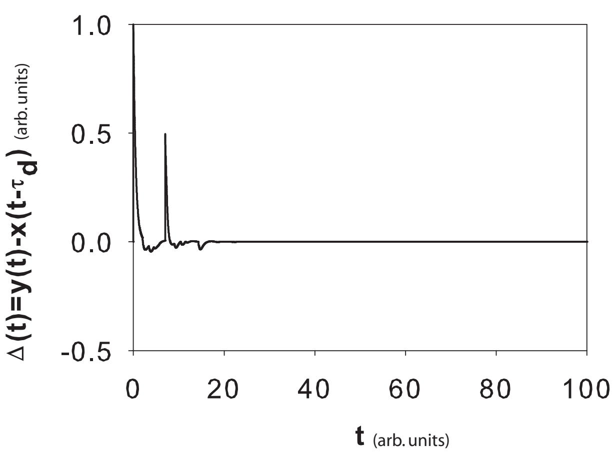
The simulation result illustrated in Fig. 12 shows that the manifold of $y(t) = x(t - 7.0)$ is established and maintained. The manifold's delay can be seen in Fig. 12(a) and Fig. 12(b). The synchronization error vanishes eventually as given in Fig. 12(c), it confirms the synchronous regime of nonidentical MTDSs.



(a) Time series of $x(t)$ and $y(t)$



(b) Portrait of $x(t-7.0)$ versus $y(t)$



(c) Synchronization error $\Delta(t) = y(t) - x(t-7.0)$

Fig. 12. Simulation result of lag synchronization of partially identical MTDSs.

Example 6:
In this example, the demonstration for lag synchronization of completely nonidentical MTDSs given in Section 3.2.2 is presented. the equations representing for the master and slave are as
Master:

$$\frac{dx}{dt} = -\alpha x + m_1 \frac{x_{\tau_1^{(M)}}}{1 + x^6_{\tau_1^{(M)}}} + m_2 \frac{x_{\tau_2^{(M)}}}{1 + x^8_{\tau_2^{(M)}}} + m_3 \frac{x_{\tau_3^{(M)}}}{1 + x^{10}_{\tau_3^{(M)}}} \tag{93}$$

Slave:

$$\begin{aligned} \frac{dy}{dt} &= -\alpha y + n_1 siny_{\tau_1^{(S)}} + n_2 siny_{\tau_2^{(S)}} + n_3 siny_{\tau_3^{(S)}} + \\ &+ n_4 siny_{\tau_4^{(S)}} + DS(t) \end{aligned} \tag{94}$$

It is clear that the sets of function forms are $S_M = \{\frac{z}{1+z^6}, \frac{z}{1+z^8}, \frac{z}{1+z^{10}}\}$, $S_S = \{sinz\}$, $S_I = S_M \cap S_S \equiv \Phi$. Thus, the subset of function form for DSG is $S_{DSG} \subseteq S_M \cup S_S =$

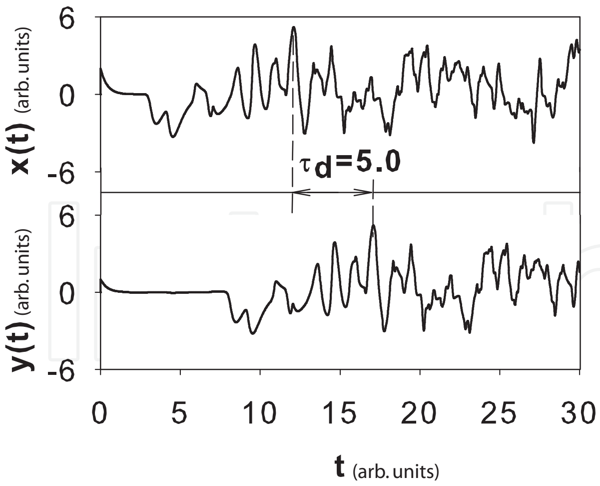
$\{sinz, \frac{z}{1+z^6}, \frac{z}{1+z^8}, \frac{z}{1+z^{10}}\}$, and the driving signal's equation must be chosen as

$$DS(t) = k_1 \frac{x_{\tau_1^{(DS)}}}{1 + x^6_{\tau_1^{(DS)}}} + k_2 \frac{x_{\tau_2^{(DS)}}}{1 + x^8_{\tau_2^{(DS)}}} + k_3 \frac{x_{\tau_3^{(DS)}}}{1 + x^{10}_{\tau_3^{(DS)}}} + \\ + k_4 sinx_{\tau_4^{(DS)}} + k_5 sinx_{\tau_5^{(DS)}} + k_6 sinx_{\tau_6^{(DS)}} + k_7 sinx_{\tau_7^{(DS)}} \tag{95}$$

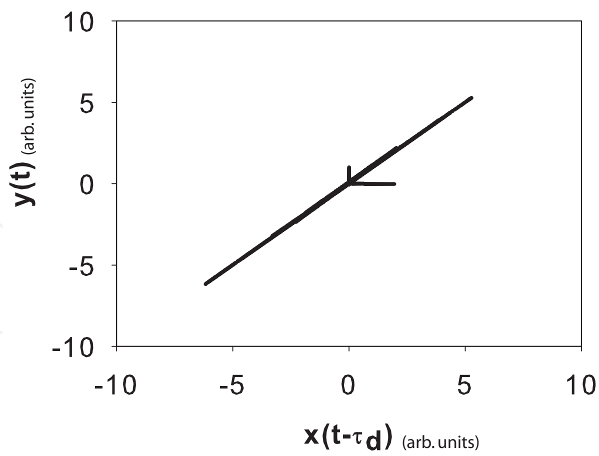
Following to the general description above, the chosen relation of delays and coefficients for the manifold of $y(t) = x(t - \tau_d)$ are as: $k_1 = m_1$, $k_2 = m_2$, $k_3 = m_3$, $k_4 = -n_1$, $k_5 = -n_2$, $k_6 = -n_3$, $k_7 = -n_4$, $\tau_1^{(DS)} = \tau_1^{(M)} + \tau_d$, $\tau_2^{(DS)} = \tau_2^{(M)} + \tau_d$, $\tau_3^{(DS)} = \tau_3^{(M)} + \tau_d$, $\tau_4^{(DS)} = \tau_1^{(S)} + \tau_d$, $\tau_5^{(DS)} = \tau_2^{(S)} + \tau_d$, $\tau_6^{(DS)} = \tau_3^{(S)} + \tau_d$, and $\tau_7^{(DS)} = \tau_4^{(S)} + \tau_d$. And, the value of parameters and delays are adopted for simulation as: $\alpha = 2.5$, $m_1 = -15.5$, $m_2 = -20.2$, $m_3 = -18.4$, $n_1 = -0.3$, $n_2 = -0.2$, $n_3 = -0.4$, $n_4 = -0.6$, $k_1 = -15.5$, $k_2 = -20.2$, $k_3 = -18.4$, $k_4 = 0.3$, $k_5 = 0.2$, $k_6 = 0.4$, $k_7 = 0.6$, $\tau_d = 5.0$, $\tau_1^{(M)} = 2.8$, $\tau_2^{(M)} = 6.4$, $\tau_3^{(M)} = 3.9$, $\tau_1^{(S)} = 1.7$, $\tau_2^{(S)} = 6.5$, $\tau_3^{(S)} = 4.1$, $\tau_4^{(S)} = 8.0$, $\tau_1^{(DS)} = 7.8$, $\tau_2^{(DS)} = 11.4$, $\tau_3^{(DS)} = 8.9$, $\tau_4^{(DS)} = 6.7$, $\tau_5^{(DS)} = 11.5$, $\tau_6^{(DS)} = 9.1$, and $\tau_7^{(DS)} = 13.0$.
Shown in Fig. 13 is the time series of state variables, the portrait of $x(t - 5.0)$ versus $y(t)$ and synchronization error $\Delta(t) = y(t) - x(t - 5.0)$, and it is easy to realize that the desired manifold is created and maintained.

5. Discussion

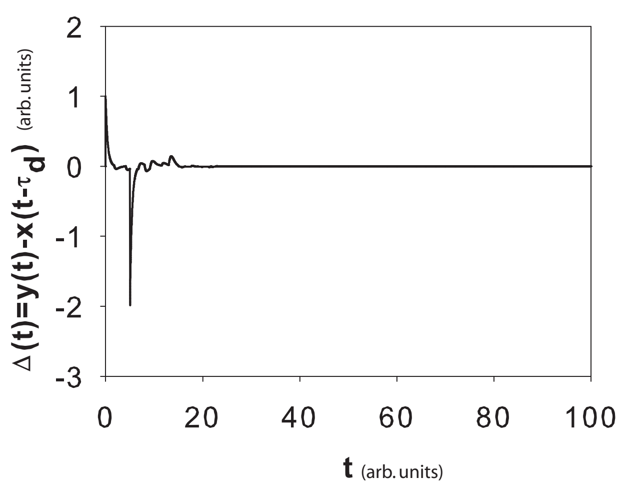
In this section, the discussion is given on four aspects, i.e., the sufficient condition for synchronization, the connection between the synchronous schemes in the proposed models, the form of driving signal and the complicated dynamics of MTDSs in compared to STDSs. These will confirm the application of the proposed synchronization model in secure communications.
Firstly, the sufficient conditions for synchronization given in Eqs. (23), (35), (43), (54), (72) and (79) are loose for adopting value of parameters and delays. It is dependent on value of parameters and not on delays since $f'(x)$ is not a piecewise function with respect to x. This allows to arrange multiple slaves being synchronized with one master at the same time.
Secondly, it is easy to realize from the connection between the synchronous schemes that transition from lag synchronization to anticipating one can be done by changing the relation between delays in DSG from $\tau_{P+j} = \tau_i + \tau_d$ to $\tau_{P+j} = \tau_i - \tau_d$ (see Eqs. (19) and (40)). Moreover, the sufficient condition for lag synchronization is identical to that for anticipating synchronization as presented in Eqs. (23) and (43). Besides, transition from lag synchronization with the synchronization manifold of $y(t) = x(t - \tau_d)$ in Eq. (15) to projective-lag synchronization with the manifold of $ay(t) = bx(t - \tau_d)$ given in Eq. (24) has been done by changing the relation between parameters from $m_i - k_j = n_i$ to $bm_i - ak_j = an_i$ (see Eqs. (21) and (33)); a, b are nonzero real numbers. Similar to the case of transition from lag synchronization to anticipating one, projective-anticipating synchronization has been achieved by changing the relation between delays in projective-lag synchronization from $\tau_{P+j} = \tau_i + \tau_d$ to $\tau_{P+j} = \tau_i - \tau_d$ (see Eqs. (19) and (40)) whereas the relation between parameters and the sufficient condition for synchronization have been kept intact (see Eqs. (33), (35) and (54)). As a special case, if the value of τ_d is set to zero, then lag and anticipating synchronization will become the scheme of complete synchronization of MTDSs and the schemes of projective-lag and projective-anticipating synchronizations turn into the



(a) Time series of $x(t)$ and $y(t)$



(b) Portrait of $x(t-5.0)$ versus $y(t)$



(c) Synchronization error $\Delta(t) = y(t) - x(t-5.0)$

Fig. 13. Simulation for lag synchronization of completely nonidentical MTDSs.

projective synchronization of MTDSs.

Thirdly, in the proposed model of identical MTDSs, it is observed that the driving signals given in Eqs. (13) and (56) are in the form of sum of nonlinear transforms, and they are commonly used for considering all the synchronous schemes. The reason for choosing such the form is to obtain synchronization error dynamics being in the linear form. Then, the Krasovskii-Lyapunov theory is applied to get sufficient condition for synchronization. Assumptions made to $f(.)$ being differentiable and bounded as well as obliged relations made to parameters and delays are also for this reason. This must be appropriate to given forms of the master and slave.

Lastly, earlier part of the paper has been mentioned the prediction that MTDSs may hold more complicated dynamics than STDSs do. This has been confirmed from the result of numerical simulation given in Section 2.2. It is well-known that Lyapunov exponents and metric entropy are measure of complexity degree for chaotic dynamics. That is, in the specific example of two-delays Mackey-Glass system, it is possible to obtain dynamics with LLE of approximate 0.7 and metric entropy of around 1.4 as shown in Fig. 6 by adopting suitable value of parameters and delays. Recall that, in the specific example of single time-delay Mackey-Glass system examined by J.D. Farmer (Farmer, 1982), LLE and metric entropy were reported at around 0.07 and 0.1, respectively. The 'V' shape of LLE and metric entropy with respect to m_1 and m_2 in Figs. 4 and 5 illustrates more intuitively. At small value of m_i, the two-delays system tends to be single time-delay system due to weak feedback. The shift of 'V' shape in the case of $m_3 = 3.0$ can be interpreted that there is some correlation to value of delays. Here, τ_2 associated with m_2 holds largest value. Undoubtedly, MTDSs holds dynamics which is more complicated than that of STDSs.

6. Conclusion

In this chapter, the synchronization model of coupled identical MTDSs has been presented, in which the coupling signal is sum of nonlinear transforms of delayed state variable. The synchronous schemes of lag, anticipating, projective-lag and projective-anticipating have been examined in the proposed models. In addition, the synchronization model of coupled nonidentical MTDS has been studied in two cases, i.e., partially identical and completely nonidentical. The scheme of lag synchronization has been used for demonstrating and verifying the cases. The simulation result has consolidated the general description to the proposed synchronous schemes. Noticeably, combination between synchronous schemes of projective and lag/anticipating is first time mentioned and investigated.

The transition between the lag and anticipating synchronization as well as between the projective-lag to projective-anticipating synchronization can be yielded simply by adjusting the relation between delays while the change from the lag to projective-lag synchronization and from the anticipating to projective-anticipating synchronization has been realized by modifying the relation between coefficients. Similarly, other synchronous schemes of coupled nonidentical MTDSs can be investigated as ways dealing in the synchronization models of identical MTDSs, and synchronous regimes will also be established as expected. This allows the synchronization models becoming flexible in selection of working scheme and switch among various schemes.

In summary, the proposed synchronization models present advantages to the application of secure communications in comparison with conventional ones. Advantages lie in both the complexity of driving signal and infinite-dimensional dynamics.

7. Acknowledgments

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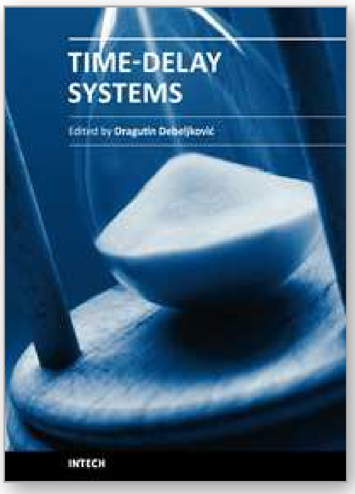

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Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, robotics, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of controllability, observability, robustness, optimization, adaptive control, pole placement and particularly stability and robustness stabilization for this class of systems, has been one of the main interests for many scientists and researchers during the last five decades.

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