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On Stable Periodic Solutions of One Time Delay System Containing Some Nonideal Relay Nonlinearities

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1. Introduction

Problems of stabilization and determining of stablility characteristics of steady-state regimes are among the central in a control theory. Especial difficulties can be met when dealing with the systems containing nonlinearities which are nonanalytic function of phase. Different models describing nonlinear effects in real control systems (e.g. servomechanisms, such as servo drives, autopilots, stabilizers etc.) are just concern this type, numerous works are devoted to the analysis of problem of stable oscillations presence in such systems.
Time delays appear in control systems frequently and are important due to significant impact on them. They affect substantially on stability properties and configuration of steady state solutions. An accurate simultaneous account of nonlinear effects and time delays allows to receive adequate models of real control systems.
This work contains some results concerning to a stability problem for periodic solutions of nonlinear controlled system containing time delay. It corresponds further development of an article: Kamachkin & Stepanov (2009). Main results obtained below might generally be put in connection with classical results of V.I. Zubov's control theory school (see Zubov (1999), Zubov & Zubov (1996)) and based generally on work Zubov & Zubov (1996).
Note that all examples presented here are purely illustrative; some examples concerning to similar systems can be found in Petrov & Gordeev (1979), Varigonda & Georgiou (2001).

2. Models under consideration

Consider a system

$$\dot{x} = Ax + cu(t-\tau), \tag{1}$$

here $x = x(t) \in \mathbb{E}^n$, $t \geq t_0 \geq \tau$, A is real $n \times n$ matrix, $c \in \mathbb{E}^n$, vector $x(t)$, $t \in [t_0 - \tau,\, t_0]$, is considered to be known. Quantity $\tau > 0$ describes time delay of actuator or observer. Control statement u is defined in the following way:

$$u(t-\tau) = f\left(\sigma(t-\tau)\right), \quad \sigma(t-\tau) = \gamma' x(t-\tau), \quad \gamma \in \mathbb{E}^n, \quad \|\gamma\| \neq 0;$$

nonlinearity f can, for example, describe a nonideal two-position relay with hysteresis:

$$f(\sigma) = \begin{cases} m_1, & \sigma < l_2, \\ m_2, & \sigma > l_1, \end{cases} \tag{2}$$

here $l_1 < l_2$, $m_1 < m_2$; and $f(\sigma(t)) = f_- = f(\sigma(t-0))$ if $\sigma \in [l_1; l_2]$.
In addition to the nonlinearity (2) a three-position relay with hysteresis will be considered:

$$f(\sigma) = \begin{cases} 0, & \begin{cases} |\sigma| \le l_0, \\ |\sigma| \in (l_0;\ l], \quad f_- = 0; \end{cases} \\ m_1, & \begin{cases} \sigma \in [-l;\ -l_0), \ f_- = m_1, \\ \sigma < -l; \end{cases} \\ m_2, & \begin{cases} \sigma \in (l_0;\ l], \quad f_- = m_2, \\ \sigma > l; \end{cases} \end{cases} \tag{3}$$

(here $m_1 < m < m_2$, $0 < l_0 < l$);
Suppose that hysteresis loops for the nonlinearities are walked around in counterclockwise direction.

3. Stability of periodic solutions

Denote $x(t - t_0, x_0, u)$ solution of the system (1) for unchanging control law u and initial conditions (t_0, x_0).
Let the system (1), (3) has a periodic solution with four switching points $\hat{s}_i$ such as

$$\hat{s}_1 = x(T_4, \hat{s}_4, m_2), \quad \hat{s}_2 = x(T_1, \hat{s}_1, 0), \quad \hat{s}_3 = x(T_2, \hat{s}_2, m_1), \quad \hat{s}_4 = x(T_3, \hat{s}_3, 0).$$

Let s_i, $i = \overline{1,4}$ are points of this solution (preceding to the corresponding $\hat{s}_i$) such as

$$\gamma' s_1 = l_0, \quad \gamma' s_2 = -l, \quad \gamma' s_3 = -l_0, \quad \gamma' s_4 = l,$$

(let us name them Ťpre-switching pointsŤ, for example), and

$$\hat{s}_1 = x(\tau, s_1, m_2), \quad \hat{s}_2 = x(\tau, s_2, 0), \quad \hat{s}_3 = x(\tau, s_3, m_1), \quad \hat{s}_4 = x(\tau, s_4, 0),$$

or

$$\hat{s}_{i+1} = x(T_i, \hat{s}_i, u_i), \quad \hat{s}_i = x(\tau, s_i, u_{i-1}),$$

where

$$u_1 = 0, \quad u_2 = m_1, \quad u_3 = 0, \quad u_4 = m_2$$

(hereafter suppose that indices are cyclic, i.e. for $i = \overline{1,m}$ one have $i + 1 = 1$ if $i = m$ and $i - 1 = m$ if $i = 1$).
Denote

$$v_i = As_{i+1} + cu_i, \quad k_i = \gamma' v_i.$$

Theorem 1. *Let $k_i \neq 0$ and $\|M\| < 1$, where*

$$M = \prod_{i=4}^{1} M_i, \quad M_i = \left(I - k_i^{-1} v_i \gamma'\right) e^{AT_i},$$

then the periodic solution under consideration is orbitally asymptotically stable.

Proof As

$$s_{i+1} = e^{A(T_i-\tau)}\hat{s}_i + \int_0^{T_i-\tau} e^{A(T_i-\tau-t)}cu_i\,dt, \quad \hat{s}_i = e^{A\tau}s_i + \int_0^{\tau} e^{A(\tau-t)}cu_{i-1}\,dt,$$

then the expression for s_{i+1} can be written in a following form:

$$\begin{aligned} s_{i+1} &= e^{AT_i}s_i + e^{AT_i}\int_0^{\tau} e^{-At}cu_{i-1}dt + \int_0^{T_i-\tau} e^{A(T_i-\tau-t)}cu_i dt = \\ &= e^{AT_i}\left(s_i + \int_0^{\tau} e^{-At}cu_{i-1}\,dt + \int_{\tau}^{T_i} e^{-At}cu_i\,dt\right). \end{aligned}$$

So,

$$(s_{i+1})'_{s_i} = e^{AT_i}, \quad (s_{i+1})'_{T_i} = As_{i+1} + cu_i = v_i,$$

and

$$\begin{aligned} d\left(\gamma' s_{i+1}\right) = 0 = \gamma' e^{AT_i}ds_i + \gamma' v_i dT_i, \quad dT_i = -k_i^{-1}\gamma' e^{AT_i}ds_i, \\ ds_{i+1} = e^{AT_i}ds_i - v_i k_i^{-1}\gamma' e^{AT_i}ds_i = \left(I - k_i^{-1}v_i\gamma'\right)e^{AT_i}\,ds_i = M_i ds_i. \end{aligned}$$

Denote ds_1^k the successive deviations of pre-switching points of some diverged solution from s_1. In such a case

$$ds_1^{k+1} = \prod_{i=4}^{1} M_i ds_1^k.$$

The system under consideration causes continuous contracting mapping of some neighbourhood of the point s_1 lying on hyperplane $s = l_0$, to itself. Use of fixed point principle (Nelepin (2002)) completes the proof. ■

Example 1. *Let* $\tau = 0.3$,

$$A = \begin{pmatrix} -0.1 & -0.1 & 0 \\ 0.1 & -0.1 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0.2 \\ 0 \\ -1 \end{pmatrix},$$

$$m_{1,2} = \mp 1, \quad l_0 = 0.1, \quad l = 0.5.$$

System (1), (3) has periodic solution with four switching points; the pre-switching points are:

$$s_1 \approx \begin{pmatrix} 0.468349 \\ 0.497302 \\ -0.006307 \end{pmatrix}, \quad s_2 \approx \begin{pmatrix} 0.005176 \\ -0.000633 \\ 0.501036 \end{pmatrix}, \quad s_3 = -s_1, \quad s_4 = -s_2;$$

and

$$T_1 \approx 53.6354, \quad T_2 \approx 0.7973, \quad T_3 = T_1, \quad T_4 = T_2.$$

As $\|M\| \approx 0.0078 < 1$, *then the periodic solution is orbitally asymptotically stable.*

Similarly, the system (1), (3) may have a periodic solution with a pair of switching points $\hat{s}_{1,2}$ and a pair of pre-switching points $s_{1,2}$ such as

$$\hat{s}_1 = x\left(T_2, \hat{s}_2, m_2\right), \quad \hat{s}_2 = x\left(T_1, \hat{s}_1, 0\right),$$
$$\hat{s}_1 = x\left(\tau, s_1, m_2\right), \quad \gamma' s_1 = l_0, \quad \hat{s}_2 = x\left(\tau, s_2, 0\right), \quad \gamma' s_1 = l.$$

for some positive constants $T_{1,2}$. This solution will be orbitally asymptotically stable if

$$k_1 = \gamma' v_{1,2} \neq 0, \quad \text{where} \quad v_i = As_j + cu_i, \quad i \neq j, \quad u_1 = 0, \quad u_2 = m_2,$$

and

$$\|M\| = \left\| \left(I - k_2^{-1} v_2 \gamma'\right) e^{AT_2} \left(I - k_1^{-1} v_1 \gamma'\right) e^{AT_1} \right\| < 1$$

(the proof is similar to the previous one).

Example 2. *Let* $\tau = 0.5$,

$$A = \begin{pmatrix} -0.1 & -0.2 & 0 \\ 0.2 & -0.1 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0.1 \\ 0 \\ -1 \end{pmatrix},$$
$$l_0 = 0.75, \quad l = 1, \quad m_{1,2} = \mp 1.$$

Then the system (1), (3) has a periodic solution with pre-switching points

$$s_1 = \begin{pmatrix} 0.2727 \\ 0.2886 \\ -0.7227 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad T_1 = 149.6021, \quad T_2 = 0.7847,$$
$$\|M\| \approx 0.9286 < 1,$$

and the solution is orbitally asymptotically stable.

4. Some extensions (bilinear system, multiple control etc.)

Consider a bilinear system

$$\dot{x} = Ax + (Cx + c)\, u(t - \tau), \tag{4}$$

In case of piecewise constant nonlinearity it is easy to obtain sufficient conditions for orbital asymptotical stability of periodic solutions of this system.
Denote $x_i(t - t_0, x_0), \ i = \overline{1,4}$ solution of the system

$$\dot{x} = A_i x + c_i,$$

where (t_0, x_0) are initial conditions and

$$A_1 = A_3 = A, \quad A_2 = A + Cm_1, \quad A_4 = A + Cm_2, \quad c_1 = c_3 = \mathbf{0}, \quad c_2 = cm_1, \quad c_4 = cm_2.$$

Lef the control u is given by (3) and the system (4), (3) has a periodic solution with four control switching points (see the Theorem 1) $\hat{s}_i$ and "pre-switching" points s_i such as

$$\hat{s}_{i+1} = x_i\left(T_i, \hat{s}_i\right), \quad \gamma' s_1 = l_0, \quad \gamma' s_2 = -l, \quad \gamma' s_3 = -l_0, \quad \gamma' s_4 = l.$$

Denote

$$v_i = A_i s_{i+1} + c_i, \quad k_i = \gamma' v_i, \quad i = \overline{1,4}.$$

Theorem 2. *If $k_i \neq 0$ and*

$$\|M\| = \left\| \prod_{i=4}^{1} \left(I - k_i^{-1} v_i \gamma' \right) e^{A_i T_i + (A_{i-1} - A_i)\tau} \right\| < 1,$$

then the periodic solution under consideration is orbitally asymptotically stable.

Proof As

$$s_{i+1} = x_i \left(T_i - \tau, \hat{s}_i \right) = x_i \left(T_i - \tau, x_{i-1} \left(\tau, s_i \right) \right) =$$
$$= e^{A_i (T_i - \tau)} \left(e^{A_{i-1}\tau} s_i + \int_0^{\tau} e^{A_{i-1}(\tau - t)} c_{i-1} dt \right) + \int_0^{T_i - \tau} e^{A_i (T_i - \tau - t)} c_i dt =$$
$$= e^{A_i T_i + (A_{i-1} - A_i)\tau} s_i + e^{A_i (T_i - \tau)} \int_0^{\tau} e^{A_{i-1}(\tau - t)} c_{i-1} dt + + \int_0^{T_i - \tau} e^{A_i (T_i - \tau - t)} c_i dt,$$

then

$$\left(s_{i+1} \right)'_{s_i} = e^{A_i T_i + (A_{i-1} - A_i)\tau}, \quad \left(s_{i+1} \right)'_{T_i} = A_i s_{i+1} + c_i.$$

So, as $d \left(\gamma' s_{i+1} \right) = 0$,

$$\gamma' e^{A_i T_i + (A_{i-1} - A_i)\tau} ds_i = -k_i dT_i, \quad ds_{i+1} = \left(I - k_i^{-1} v_i \gamma' \right) e^{A_i T_i + (A_{i-1} - A_i)\tau} ds_i,$$

and $ds_1^{k+1} = M ds_1^k$. Use of fixed point principle completes the proof. ■

Example 3. *Let, for example, $\tau = 0.3$,*

$$A = \begin{pmatrix} -0.1 & -0.05 & 0 \\ 0.1 & -0.05 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0.05 & 0 \\ 0.05 & -0.1 & 0.05 \\ 0 & -0.05 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$
$$\gamma' = \begin{pmatrix} -0.2 & 0.5 & -1 \end{pmatrix}, \quad l_0 = 0.1, \quad l = 0.5, \quad m_{1,2} = \mp 1.$$

In such a case the system (4), (3) has periodic solution with pre-switching points

$$s_1 \approx \begin{pmatrix} 0.6819 \\ 0.5383 \\ 0.0328 \end{pmatrix}, \quad s_2 \approx \begin{pmatrix} -0.0534 \\ -0.0073 \\ 0.5070 \end{pmatrix}, \quad s_3 \approx \begin{pmatrix} -0.6096 \\ -0.6396 \\ -0.0979 \end{pmatrix}, \quad s_4 \approx \begin{pmatrix} 0.1127 \\ -0.0664 \\ -0.5557 \end{pmatrix},$$
$$T_1 \approx 42.2723, \quad T_2 \approx 0.8977, \quad T_3 \approx 33.5405, \quad T_4 \approx 0.8969.$$

One can verify that $k_i \neq 0$, and

$$\|M\| \approx 0.8223 < 1.$$

So, the solution under consideration is orbitally asymptotically stable.

Note that if matrices $A_{1,2} = A + Cm_{1,2}$ are Hurwitz, and

$$-\gamma' A_2^{-1} c m_2 < l_1, \quad -\gamma' A_1^{-1} c m_1 > l_2,$$

then the system (4), (2) has at least one periodic solution.
By the analogy with the system (1), a system with multiple controls can be observed:

$$\dot{x} = Ax + c_1 u_1 \left(\sigma_1 (t - \tau_1) \right) + c_2 u_2 \left(\sigma_2 (t - \tau_2) \right). \tag{5}$$

Suppose for simplicity that u_i are simple hysteresis nonlinearities given by (2):

$$u_i(\sigma) = u(\sigma) = \begin{cases} m_1, & \sigma_i < l_2, \\ m_2, & \sigma_i > l_1, \end{cases} \quad \sigma_i = \gamma_i' x, \quad i = 1, 2.$$

Denote $x\left(t - t_0, x_0, u_1, u_2\right)$ solution of the system (5) for unchanging control laws $u_{1,2}$ and initial conditions (t_0, x_0). Let the system has periodic solution with four switching $(\hat{s}_i)$ and pre-switching (s_i) points such as

$$\hat{s}_1 = x(T_4, \hat{s}_4, m_2, m_2), \quad \hat{s}_2 = x(T_1, \hat{s}_1, m_1, m_2), \quad \hat{s}_3 = x(T_2, \hat{s}_2, m_1, m_1), \quad \hat{s}_4 = x(T_3, \hat{s}_3, m_2, m_1),$$
$$\hat{s}_1 = x(\tau, s_1, m_2, m_2), \quad \hat{s}_2 = x(\tau, s_2, m_1, m_2), \quad \hat{s}_3 = x(\tau, s_3, m_1, m_1), \quad \hat{s}_4 = x(\tau, s_4, m_2, m_1),$$
$$\gamma_1' s_1 = -l_1, \quad \gamma_2' s_2 = -l_2, \quad \gamma_1' s_3 = l_1, \quad \gamma_2' s_4 = l_2.$$

Denote

$$p_1 = c_1 m_1 + c_2 m_2, \quad p_2 = c_1 m_1 + c_2 m_1, \quad p_3 = c_1 m_2 + c_2 m_1, \quad p_4 = c_1 m_2 + c_2 m_2,$$
$$v_i = A s_{i+1} + p_i, \quad i = \overline{1,4}, \quad k_1 = \gamma_2' v_1, \quad k_2 = \gamma_1' v_2, \quad k_3 = \gamma_2' v_3, \quad k_4 = \gamma_1' v_4,$$
$$M_1 = \left(I - k_1^{-1} v_1 \gamma_2'\right) e^{AT_1}, \quad M_2 = \left(I - k_2^{-1} v_2 \gamma_1'\right) e^{AT_2},$$
$$M_3 = \left(I - k_3^{-1} v_3 \gamma_2'\right) e^{AT_3}, \quad M_4 = \left(I - k_4^{-1} v_4 \gamma_1'\right) e^{AT_4}.$$

It is easy to verify that the solution under consideration is orbitally asymptotically stable if $k_i \neq 0$ and

$$\left\| \prod_{i=4}^{1} M_i \right\| < 1.$$

Example 4. *Consider a trivial case:*

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix}.$$

So the system can be rewritten as a pair of independent equations

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + u\left(\alpha_1 x(t - \tau_1)\right), \\ \dot{x}_2 = \lambda_2 x_2 + u\left(\alpha_2 x(t - \tau_2)\right); \end{cases}$$

or

$$\begin{cases} \dot{\sigma}_1 = \lambda_1 \sigma_1 + \alpha_1 u\left(\sigma_1(t - \tau_1)\right), \\ \dot{\sigma}_2 = \lambda_2 \sigma_2 + \alpha_2 u\left(\sigma_2(t - \tau_2)\right). \end{cases}$$

Let, for example, $\lambda_1 > 0$, $\lambda_2 < 0$, $l_1 = -l_2 = -l$, $m_1 = -m_2 = -m$, $\tau_1 = \tau_2 = \tau$. *Denote*

$$\hat{l}_i = e^{\lambda_i \tau} l - \alpha_i \lambda_i^{-1} \left(e^{\lambda_i \tau} - 1\right) m, \quad i = 1, 2.$$

Between switchings σ *looks as follows:*

$$\sigma_i(t) = e^{\lambda_i t} \sigma_i(0) + \alpha_i \lambda_i^{-1} \left(e^{\lambda_i t} - 1\right) u, \quad i = 1, 2.$$

Suppose t_1 is a positive constant such as

$$\sigma_1(0) = -\hat{l}_1, \quad \sigma_1(0.5t_1) = \hat{l}_1, \quad u = -m;$$

i.e.

$$\frac{\alpha_1 m}{\lambda_1} - \hat{l}_1 = \left(\frac{\alpha_1 m}{\lambda_1} + \hat{l}_1\right) e^{0.5\lambda_1 t_1}, \quad t_1 = \frac{2}{\lambda_1} \ln \frac{\alpha_1 m - \lambda_1 \hat{l}_1}{\alpha_1 m + \lambda_1 \hat{l}_1}.$$

Similarly,

$$t_2 = \frac{2}{\lambda_2} \ln \frac{\alpha_2 m - \lambda_2 \hat{l}_2}{\alpha_2 m + \lambda_2 \hat{l}_2}.$$

If t_i are commensurable quantities (i.e. t_1/t_2 is rational number) then the system has a periodic solution with the period $T = LCM(t_1, t_2)$.
This example also demonstrates that there can exist an almost periodic solution of the system (5) (as a superposition of two periodic solutions with incommensurable periods) if $t_1/t_2 \in \mathcal{I}$.
Let, for example,

$$\tau = 0.1, \quad \lambda_1 = -\lambda_2 = \lambda = 0.1, \quad l = m = 1.$$

Let us choose parameters $\alpha_{1,2}$ in such a way that $t_1 = t_2$. It is easy to verify that the latest equality holds true if

$$\frac{\alpha_1 - \lambda \hat{l}_1}{\alpha_1 + \lambda \hat{l}_1} = \frac{\alpha_2 - \lambda \hat{l}_2}{\alpha_2 + \lambda \hat{l}_2}, \quad or \quad \frac{\alpha_1}{\alpha_2} = \frac{\hat{l}_1}{\hat{l}_2}$$

So,

$$\alpha_2 = \frac{\alpha_1 \lambda l}{(\lambda l - \alpha_1 m)\, e^{2\lambda\tau} + 2\alpha_1 m e^{\lambda\tau} - \alpha_1 m}.$$

Let $\alpha_1 = -1$, then

$$\alpha_2 \approx -0.979229,$$

then we can calculate $\hat{l}_{1,2}$:

$$\hat{l}_1 \approx 1.110552, \quad \hat{l}_2 \approx 1.087485.$$

And, finally,

$$t_1 = t_2 \approx 4.460606.$$

The system under consideration has a T-periodic solution, $T = t_i$. Let $s_1' = \begin{pmatrix}1 & 0\end{pmatrix}$, then

$$s_2' \approx \begin{pmatrix}0.19809 & 1.02122\end{pmatrix}, \quad s_3 = -s_1, \quad s_4 = -s_4,$$
$$T_1 = T_3 \approx 1.07715, \quad T_2 = T_4 \approx 1.15315;$$

and

$$ds_1^{k+1} = M ds_1^k, \quad M = \begin{pmatrix} 0 & 0 \\ 1.1362... & 1 \end{pmatrix}.$$

So, as $s_{1,1} = 1$, then $ds_{1,1} = 0$,

$$ds_{1,2}^{k+1} = ds_{1,2}^k,$$

and the periodic solution under consideration cannot be asymptotically stable (of course this fact can be established from other general considerations).
It is obvious that the system under consideration may have periodic solutions with greater amount of switching points (depending of $LCM(t_1, t_2)$ value).

Similar computations can be observed in case of nonlinearity (3).

5. Stability in case of multiple delays

In more general case the system under consideration can also contain several nonlinearities or several positive delays τ_i ($i = \overline{1,k}$) in control loop:

$$\dot{x}(t) = Ax(t) + c\,f\left(\sum_{i=1}^{k} \gamma_i' x(t-\tau_i)\right), \quad \gamma_i \in \mathbb{E}^n, \quad \|\gamma_i\| \neq 0. \tag{6}$$

Let, for example, $k = 2$, $\tau_1 = 0$, $\tau_2 = \tau$, denote $\hat{\gamma} = \gamma_1$, $\gamma = \gamma_2$, i.e.

$$\dot{x}(t) = Ax(t) + cf\left(\hat{\sigma}(t) + \sigma(t-\tau)\right), \quad \hat{\sigma} = \hat{\gamma}'x, \quad \sigma = \gamma' x. \tag{7}$$

Consider one simple particular case. Let f is given by the (2) and the system (7), (2) has a periodic solution with two switching points $\hat{s}_{1,2}$ such as

$$\hat{s}_1 = x(T_2, \hat{s}_2, m_2), \quad \hat{s}_2 = x(T_1, \hat{s}_1, m_1),$$
$$\hat{\gamma}'\hat{s}_1 + \gamma' s_1 = l_1, \quad \hat{\gamma}'\hat{s}_2 + \gamma' s_2 = l_2.$$

Here

$$\hat{s}_2 = e^{A\tau}s_2 + \int_0^\tau e^{A(\tau-t)}cm_1 dt, \quad \hat{s}_1 = e^{A\tau}s_1 + \int_0^\tau e^{A(\tau-t)}cm_2 dt.$$

Denote

$$\Gamma = \left(e^{A\tau}\right)'\hat{\gamma} + \gamma, \qquad \hat{l}_1 = l_1 - \hat{\gamma}' \int_0^\tau e^{A(\tau-t)}cm_2 dt, \qquad \hat{l}_2 = l_2 - \hat{\gamma}' \int_0^\tau e^{A(\tau-t)}cm_1 dt.$$

then

$$\Gamma' s_1 = \hat{l}_1, \quad \Gamma' s_2 = \hat{l}_2.$$

Theorem 3. *Let*

$$v_1 = As_2 + cm_1, \quad v_2 = As_1 + cm_2, \quad k_{1,2} = \Gamma' v_{1,2} \neq 0,$$

and

$$\left\| \left(I - k_2^{-1} v_2 \Gamma'\right) e^{AT_2} \left(I - k_1^{-1} v_1 \Gamma'\right) e^{AT_1} \right\| < 1,$$

then the periodic solution under consideration is orbitally asymptotically stable.

Proof The proof is similar to the previous proofs. As $d\left(\Gamma' s_i\right) = 0$, then

$$ds_{i+1} = \left(I - k_I^{-1} v_i \Gamma'\right) e^{AT_i} ds_i = M_i ds_i.$$

So, $ds_1^{k+1} = M_2 M_1 ds_1^k$, and use of fixed point principle completes the proof. ∎

Note that here we can obtain sufficient conditions for the orbital stability in the alternative way. Suppose

$$\Gamma = \hat{\gamma} + \left(e^{-A\tau}\right)'\gamma, \quad \hat{l}_1 = l_1 + \gamma' \int_0^\tau e^{-At}cm_2 dt, \quad \hat{l}_2 = l_2 + \gamma' \int_0^\tau e^{-At}cm_1 dt,$$
$$v_1 = A\hat{s}_2 + cm_1, \quad v_2 = A\hat{s}_1 + cm_2, \quad k_{1,2} = \Gamma' v_{1,2},$$

in such a case

$$\Gamma' \hat{s}_i = \hat{l}_i, \quad i = 1, 2,$$

and the periodic solution will be orbitally asymptotically stable if $k_{1,2} \neq 0$ and

$$\left\| \left(I - k_2^{-1} v_2 \Gamma'\right) e^{AT_2} \left(I - k_1^{-1} v_1 \Gamma'\right) e^{AT_1} \right\| < 1.$$

All the above statements we can reformulate in a similar way, defining the above vector Γ, considering the switching points instead of pre-switching and re-defining threshold values l_i (or l_0, l in case of (3)).

Let us return to the system (6). In general case we can repeate the previous derivations. Let it has a periodic solution with two control switching points $\hat{s}_{1,2}$, such as

$$\sum_{i=1}^{k} \gamma_i' s_{1,i} = l_1, \quad \sum_{i=1}^{k} \gamma_i' s_{2,i} = l_2,$$

where

$$\hat{s}_1 = x\left(\tau_i, s_{1,i}, m_2\right), \quad \hat{s}_2 = x\left(\tau_i, s_{2,i}, m_1\right), \quad i = \overline{1,k}.$$

Then

$$\sum_{i=1}^{k} \gamma_i' \left(e^{-A\tau_i} \hat{s}_1 - \int_0^{\tau_i} e^{-At} c m_2 dt \right) = l_1, \quad \sum_{i=1}^{k} \gamma_i' \left(e^{-A\tau_i} \hat{s}_2 - \int_0^{\tau_i} e^{-At} c m_1 dt \right) = l_2,$$

and

$$\Gamma \hat{s}_j = \hat{l}_j, \quad j = 1, 2,$$

here

$$\Gamma = \sum_{i=1}^{k} \left(e^{-A\tau_i}\right)' \gamma_i, \quad \hat{l}_1 = l_1 + \sum_{i=1}^{k} \gamma_i' \int_0^{\tau_i} e^{-At} c m_2 dt, \quad \hat{l}_2 = l_2 + \sum_{i=1}^{k} \gamma_i' \int_0^{\tau_i} e^{-At} c m_1 dt.$$

So the considered periodic solution will be orbitally asymptotically stable if $k_{1,2} \neq 0$ and

$$\left\| \left(I - k_2^{-1} v_2 \Gamma'\right) e^{AT_2} \left(I - k_1^{-1} v_1 \Gamma'\right) e^{AT_1} \right\| < 1,$$

where

$$v_1 = A\hat{s}_2 + cm_1, \quad v_2 = A\hat{s}_1 + cm_2, \quad k_{1,2} = \Gamma' v_{1,2}.$$

Of course the system considered can have periodic solutions with amount of control switching points larger then two. Consider an example:

Example 5. *Consider the system (6), (2). Let $\tau_1 = 0.013$, $\tau_2 = 0.015$,*

$$A = \begin{pmatrix} -0.25 & -1. & -0.25 \\ 0.75 & 1. & 0.75 \\ 0.25 & -7. & -3.75 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0.536 \\ 0 \\ 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 \\ -1.108 \\ -0.567 \end{pmatrix},$$

$$m_{1,2} = \mp 1, \quad l_1 = -0.1, \quad l_2 = 0.5.$$

System (1), (2) has periodic solution with six switching points:

$$\hat{s}_1 \approx \begin{pmatrix} 0.69484 \\ -0.64902 \\ 2.12876 \end{pmatrix}, \quad \hat{s}_2 \approx \begin{pmatrix} 0.06226 \\ -1.91945 \\ 2.92801 \end{pmatrix}, \quad \hat{s}_3 \approx \begin{pmatrix} 0.72238 \\ -1.05935 \\ 2.95759 \end{pmatrix},$$

$$\hat{s}_4 \approx \begin{pmatrix} 0.51706 \\ -1.95858 \\ 3.43423 \end{pmatrix}, \quad \hat{s}_5 \approx \begin{pmatrix} 1.08072 \\ -0.87355 \\ 2.93260 \end{pmatrix}, \quad \hat{s}_6 \approx \begin{pmatrix} 0.11909 \\ -1.44650 \\ 2.05635 \end{pmatrix},$$

$$T_1 \approx 1.8724, \quad T_2 \approx 0.4018, \quad T_3 \approx 6.8301, \quad T_4 \approx 0.4019, \quad T_5 \approx 1.6087, \quad T_6 \approx 0.4084.$$

Let

$$\Gamma = \left(e^{-A\tau_1}\right)' \gamma_1 + \left(e^{-A\tau_2}\right)' \gamma_2 \approx \left(0.552607 \ -1.144496 \ -0.584956\right),$$
$$\hat{l}_1 = l_1 + \gamma_1' \int_0^{\tau_1} e^{-At} c m_2 dt + \gamma_2' \int_0^{\tau_2} e^{-At} c m_2 dt \approx -0.118450,$$
$$\hat{l}_2 = l_2 + \gamma_1' \int_0^{\tau_1} e^{-At} c m_1 dt + \gamma_2' \int_0^{\tau_2} e^{-At} c m_1 dt \approx 0.518450,$$

then

$$\Gamma' \hat{s}_1 = \Gamma' \hat{s}_3 = \hat{l}_1, \quad \Gamma' \hat{s}_2 = \Gamma' \hat{s}_4 = \hat{l}_1.$$

Denote

$$u_{2k+1} = m_1, \quad u_{2k} = m_2.$$

One can verify that

$$k_i = \Gamma' \left(A\hat{s}_{i+1} + cu_i\right) \neq 0, \quad i = \overline{1,6}.$$

Let

$$M_i = \left(I - k_i^{-1} \left(As_{i+1} + cu_i\right) \Gamma'\right) e^{AT_i},$$

in such a case

$$\|M\| = \left\| \prod_{i=6}^{1} M_i \right\| \approx 0.13771 < 1$$

and the periodic solution under consideration is asymptotically orbitally stable.

Let us obtain similar results for the system (4). Suppose for simplicity that

$$\dot{x} = Ax + (Cx + c) f\left(\hat{\sigma}(t) + \sigma(t-\tau)\right), \quad \hat{\sigma} = \hat{\gamma}' x, \quad \sigma = \gamma' x. \tag{8}$$

Let f is given by the (2). Denote

$$A_i = A + Cm_i, \quad c_i = cm_i, \quad i = 1,2, \quad x_i(T, x_0) = e^{A_i T} x_0 + \int_0^T e^{A_i(T-t)} c_i dt.$$

Let the system (8), (2) has a periodic solution with two switching points $\hat{s}_{1,2}$ such as

$$\hat{s}_1 = x_2(T_2, \hat{s}_2), \quad \hat{s}_2 = x_1(T_1, \hat{s}_1),$$
$$\hat{\gamma}' \hat{s}_1 + \gamma' s_1 = l_1, \quad \hat{\gamma}' \hat{s}_2 + \gamma' s_2 = l_2,$$

here

$$\hat{s}_1 = e^{A_2\tau} s_1 + \int_0^{\tau} e^{A_2(\tau-t)} c_2 dt, \quad \hat{s}_2 = e^{A_1\tau} s_2 + \int_0^{\tau} e^{A_1(\tau-t)} c_1 dt.$$

So,

$$\hat{\gamma}' e^{A_2\tau} s_1 + \hat{\gamma}' \int_0^{\tau} e^{A_2(\tau-t)} c_2 dt + \gamma' s_1 = l_1, \quad \hat{\gamma}' e^{A_1\tau} s_2 + \hat{\gamma}' \int_0^{\tau} e^{A_1(\tau-t)} c_1 dt + \gamma' s_2 = l_2,$$

or

$$\Gamma_1' s_1 = \hat{l}_1, \quad \Gamma_2' s_2 = \hat{l}_2,$$

where

$$\Gamma_1 = \left(e^{A_2\tau}\right)'\hat{\gamma} + \gamma, \quad \Gamma_2 = \left(e^{A_1\tau}\right)'\hat{\gamma} + \gamma,$$
$$\hat{l}_1 = l_1 - \hat{\gamma}' \int_0^{\tau} e^{A_2(\tau-t)} c_2 dt, \quad \hat{l}_2 = l_2 - \hat{\gamma}' \int_0^{\tau} e^{A_1(\tau-t)} c_1 dt.$$

Let

$$v_1 = A_1 s_2 + c_1, \quad v_2 = A_2 s_1 + c_2, \quad k_1 = \Gamma_2' v_1, \quad k_2 = \Gamma_1' v_2.$$

Theorem 4. *If* $k_{1,2} \neq 0$ *and*

$$\left\| \left(I - k_2^{-1} v_2 \Gamma_1'\right) e^{A_2 T_2 + (A_1 - A_2)\tau} \left(I - k_1^{-1} v_1 \Gamma_2'\right) e^{A_1 T_1 + (A_2 - A_1)\tau} \right\| < 1,$$

where

$$A_i = A + C m_i, \quad c_i = c m_i, \quad i = 1, 2.$$

Then the considered periodic solution is orbitally asymptotically stable.

Proof As

$$s_2 = x_1\left(T_1 - \tau, x_2(\tau, s_1)\right) =$$
$$= e^{A_1 T_1 + (A_2 - A_1)\tau} s_1 + e^{A_1(T_1 - \tau)} \int_0^{\tau} e^{A_2(\tau - t)} c_2 dt + \int_0^{T_1 - \tau} e^{A_1(T_1 - \tau - t)} c_1 dt,$$
$$(s_2)'_{s_1} = e^{A_1 T_1 + (A_2 - A_1)\tau}, \quad (s_2)'_{T_1} = A_1 s_2 + c_1 = v_1,$$

then

$$0 = d\left(\Gamma_2' s_2\right) = \Gamma_2' e^{A_1 T_1 + (A_2 - A_1)\tau} ds_1 + k_1 dT_1,$$
$$dT_1 = -k_1^{-1} \Gamma_2' e^{A_1 T_1 + (A_2 - A_1)\tau} ds_1, \quad \text{and} \quad ds_2 = \left(I - k_1^{-1} v_1 \Gamma_2'\right) e^{A_1 T_1 + (A_2 - A_1)\tau} ds_1.$$

Similarly,

$$ds_2 = \left(I - k_2^{-1} v_2 \Gamma_1'\right) e^{A_2 T_2 + (A_1 - A_2)\tau} ds_2.$$

In order to finalize the proof one can use the fixed point principle for s_1. ■

In case of the system (8), (3) the sufficient conditions for orbital stability will change slightly. Let the system has periodic solution with four control switching points $\hat{s}_i$, $i = \overline{1,4}$, where

$$\hat{s}_{i+1} = x_i\left(T_1, \hat{s}_i\right).$$

Let s_i, $i = \overline{1,4}$, are points on the trajectory of the solution such as

$$\hat{s}_i = x_{i-1}\left(s_i, \tau\right),$$

and

$$\hat{\gamma}' \hat{s}_i + \gamma' s_i = l_i, \quad l_1 = l_0, \quad l_2 = -l, \quad l_3 = -l_0, \quad l_4 = l.$$

In such a case

$$\hat{\gamma}' e^{A_{i-1}\tau} s_i + \hat{\gamma}' \int_0^{\tau} e^{A_{i-1}(\tau - t)} c_{i-1} dt + \gamma' s_i = \hat{l}_i,$$

or

$$\Gamma_i s_i = \hat{l}_i, \quad i = \overline{1,4}, \quad \Gamma_i = \left(e^{A_{i-1}\tau}\right)'\hat{\gamma} + \gamma, \quad \hat{l}_i = l_i - \hat{\gamma}' \int_0^{\tau} e^{A_{i-1}(\tau - t)} c_{i-1} dt.$$

Denote

$$v_i = A_i s_{i+1} + c_i, \quad k_i = \Gamma'_{i+1} v_i, \quad M_i = \left(I - k_i^{-1} v_i \Gamma'_{i+1}\right) e^{A_i T_i + (A_{i-1} - A_i)\tau}$$

Theorem 5. *Let* $k_i \neq 0$, $i = \overline{1,4}$, *and*

$$\left\| \prod_{i=4}^{1} M_i \right\| < 1, \tag{9}$$

then the periodic solution is orbitally asymptotically stable.

Let us skip the proof, it is similar to the above one.

Example 6. *Let* A, c, $l_{1,2}$, $m_{1,2}$ *are the same as in the example 5,*

$$C = \begin{pmatrix} -0.01 & 0 & 0 \\ 0 & 0.005 & 0 \\ -0.01 & 0.01 & 0.005 \end{pmatrix},$$

and

$$\dot{x} = Ax + (Cx + c) f\left(-0.565 x_3(t) - 1.11 x_2(t - 0.015) + 0.54 x_1(t - 0.1)\right),$$

where f *is given by the (2). I.e.*

$$\tau_1 = 0, \quad \tau_2 = 0.015, \quad \tau_3 = 0.1,$$
$$\gamma'_1 = \begin{pmatrix} 0 & 0 & -0.565 \end{pmatrix}, \quad \gamma'_2 = \begin{pmatrix} 0 & -1.11 & 0 \end{pmatrix}, \quad \gamma'_3 = \begin{pmatrix} 0.54 & 0 & 0 \end{pmatrix}.$$

In such a case the system has a periodic solution with four switching points

$$\hat{s}'_1 \approx \begin{pmatrix} 1.1250 & -1.0662 & 3.3411 \end{pmatrix}, \quad \hat{s}'_2 \approx \begin{pmatrix} 0.1806 & -1.3848 & 2.0040 \end{pmatrix},$$
$$\hat{s}'_3 \approx \begin{pmatrix} 0.7081 & -0.6317 & 2.0672 \end{pmatrix}, \quad \hat{s}'_4 \approx \begin{pmatrix} 0.5502 & -2.1717 & 3.9062 \end{pmatrix},$$
$$T_1 \approx 1.5668, \quad T_2 \approx 0.3846, \quad T_3 \approx 4.4353, \quad T_4 \approx 0.3890.$$

Denote

$$A_{1,2} = A + C m_{1,2},$$
$$\Gamma_1 = \gamma_1 + \left(e^{-A_2 \tau_2}\right)' \gamma_2 + \left(e^{-A_2 \tau_3}\right)' \gamma_3 \approx \begin{pmatrix} 0.564337 & -1.035933 & -0.538052 \end{pmatrix}',$$
$$\Gamma_2 = \gamma_1 + \left(e^{-A_1 \tau_2}\right)' \gamma_2 + \left(e^{-A_1 \tau_3}\right)' \gamma_3 \approx \begin{pmatrix} 0.563215 & -1.036110 & -0.538057 \end{pmatrix}',$$

$$\hat{l}_1 = l_1 + \gamma'_2 \int_0^{\tau_2} e^{-A_2 t} c m_2 dt + \gamma'_3 \int_0^{\tau_3} e^{-A_2 t} c m_2 dt \approx -0.058212,$$
$$\hat{l}_2 = l_2 + \gamma'_2 \int_0^{\tau_2} e^{-A_1 t} c m_1 dt + \gamma'_3 \int_0^{\tau_3} e^{-A_1 t} c m_1 dt \approx 0.458270.$$

Then

$$\Gamma'_1 \hat{s}_1 = \Gamma'_1 \hat{s}_3 = \hat{l}_1, \quad \Gamma'_2 \hat{s}_2 = \Gamma'_2 \hat{s}_4 = \hat{l}_2.$$

Let

$$v_1 = A_1 \hat{s}_2 + c m_1, \quad v_2 = A_2 \hat{s}_3 + c m_2, \quad v_3 = A_1 \hat{s}_4 + c m_1, \quad v_4 = A_2 \hat{s}_1 + c m_2,$$

One can easy verify that

$$k_1 = \Gamma_2' v_1 \neq 0, \quad k_2 = \Gamma_1' v_2 \neq 0, \quad k_3 = \Gamma_2' v_3 \neq 0, \quad k_4 = \Gamma_1' v_4 \neq 0.$$

Denote

$$M_1 = \left(I - k_1^{-1} v_1 \Gamma_2'\right) e^{A_1 T_1}, \quad M_2 = \left(I - k_2^{-1} v_2 \Gamma_1'\right) e^{A_2 T_2},$$
$$M_3 = \left(I - k_3^{-1} v_3 \Gamma_2'\right) e^{A_1 T_3}, \quad M_4 = \left(I - k_4^{-1} v_4 \Gamma_1'\right) e^{A_2 T_4}.$$

and

$$\|M\| = \left\| \prod_{i=1}^{4} M_i \right\| \approx 0.3033 < 1.$$

So, as $ds_1^{k+1} = M ds_1^k$, the periodic solution under consideration is orbitally asymptotically stable.
Similar results can be obtained in case of nonlinearity (3).

6. Perturbed system

Consider a system:

$$\dot{x} = Ax + c\big(\varphi(t) + u(t-\tau)\big), \tag{10}$$

where $\varphi(t)$ is scalar T_φ-periodic continuous function of time. Let f is given by (3).
Consider a special case of the previous system (see Nelepin (2002), Kamachkin & Shamberov (1995)). Let $n = 2$,

$$\ddot{y} + g_1 \dot{y} + g_2 y = u(t-\tau) + \varphi(t), \tag{11}$$

here $y(t) \in \mathbb{R}$ is sought-for time variable, $g_{1,2}$ are real constants, $\sigma = \alpha_1 y + \alpha_2 \dot{y}$, $\alpha_{1,2}$ are real constants. Let us rewrite system (11) in vector form. Denote $z' = \begin{pmatrix} y & \dot{y} \end{pmatrix}$, in that case

$$\dot{z} = Pz + q\left(\varphi(t) + u(t-\tau)\right), \tag{12}$$
$$u(t-\tau) = f\left(\sigma(t-\tau)\right), \quad \sigma = \alpha' z,$$

where

$$P = \begin{pmatrix} 0 & 1 \\ -g_2 & -g_1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Suppose that characteristic determinant $D(s) = \det(P - sI)$ has real simple roots $\lambda_{1,2}$, and vectors q, Pq are linearly independent. In that case system (12) may be reduced to the form (10), where

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

by means of nonsingular linear transformation

$$z = Tx, \qquad T = \begin{pmatrix} \frac{N_1(\lambda_1)}{D'(\lambda_1)} & \frac{N_1(\lambda_2)}{D'(\lambda_2)} \\ \frac{N_2(\lambda_1)}{D'(\lambda_1)} & \frac{N_2(\lambda_2)}{D'(\lambda_2)} \end{pmatrix}, \quad D'(\lambda_j) = \frac{d}{ds} D(s)\bigg|_{s=\lambda_j}, \quad N_j(s) = \sum_{i=1}^{2} q_i D_{ij}(s), \tag{13}$$

$D_{ij}(s)$ is algebraic supplement for element lying in the intersection of i-th row and j-th column of determinant $D(s)$.

Note that

$$\sigma = \gamma' x, \quad \gamma = T' \alpha.$$

Furthermore, since

$$\gamma_i = -\left(D'\left(\lambda_i\right)\right)^{-1} \sum_{j=1}^{2} \alpha_j N_j\left(\lambda_i\right), \quad i = 1, 2.$$

then

$$\gamma_1 = \left(\lambda_1 - \lambda_2\right)^{-1}\left(\alpha_1 + \alpha_2\lambda_1\right), \quad \gamma_2 = \left(\lambda_2 - \lambda_1\right)^{-1}\left(\alpha_1 + \alpha_2\lambda_2\right).$$

Transformation (13) leads to the following system:

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + f\left(\sigma(t-\tau)\right) + \varphi(t), \\ \dot{x}_2 = \lambda_2 x_2 + f\left(\sigma(t-\tau)\right) + \varphi(t). \end{cases} \tag{14}$$

If, for example,

$$\alpha_1 = -\lambda_1\alpha_2,$$

then

$$\gamma_1 = 0, \quad \gamma_2 = \alpha_2, \quad \sigma = \gamma_2 x_2.$$

Function f in that case is independent of variable x_1, and

$$\dot{\sigma} = \lambda_2\sigma + \gamma_2\left(f(\gamma_2 x_2(t-\tau)) + \varphi(t)\right).$$

Solution of the latest equation when $f = u$ (where $u = m_1, m_2$ or 0) has the following form:

$$\sigma\left(t,\, t_0,\, \sigma_0,\, u\right) = e^{\lambda_2(t-t_0)}\sigma_0 + \gamma_2\, e^{\lambda_2 t}\int_{t_0}^{t} e^{-\lambda_2 s}\Big(u + \varphi(s)\Big)\, ds.$$

Let us trace out necessary conditions for existing of periodic solution of the system (10), (3) having four switching points $\hat{s}_i$:

$$\begin{aligned} \sigma_2 &= \sigma\left(t_1, t_0+\tau, \hat{\sigma}_1, 0\right), & \hat{\sigma}_2 &= \sigma\left(t_1+\tau, t_1, \sigma_2, 0\right), \\ \sigma_3 &= \sigma\left(t_2, t_1+\tau, \hat{\sigma}_2, m_1\right), & \hat{\sigma}_3 &= \sigma\left(t_2+\tau, t_2, \sigma_3, m_1\right), \\ \sigma_4 &= \sigma\left(t_3, t_2+\tau, \hat{\sigma}_3, 0\right), & \hat{\sigma}_4 &= \sigma\left(t_3+\tau, t_3, \sigma_4, 0\right), \\ \sigma_1 &= \sigma\left(t_4, t_3+\tau, \hat{\sigma}_4, m_2\right), & \hat{\sigma}_1 &= \sigma\left(t_4+\tau, t_4, \sigma_1, m_2\right), \end{aligned}$$

for some positive T_i, $i = \overline{1,4}$, and $t_i = t_{i-1} + T_i$. Denote $u_1 = 0, \quad u_2 = m_1, \quad u_3 = 0, \quad u_4 = m_2$, then

$$\begin{aligned} \sigma_{i+1} &= \sigma\left(t_i,\, t_{i-1}+\tau,\, \sigma\left(t_{i-1}+\tau,\, t_{i-1},\, \sigma_i,\, u_{i-1}\right),\, u_i\right) = \\ &= e^{\lambda_2(T_i-\tau)}\left(e^{\lambda_2\tau}\sigma_i + \gamma_2 e^{\lambda_2(t_{i-1}+\tau)}\int_{t_{i-1}}^{t_{i-1}+\tau} e^{-\lambda_2 t}\left(u_{i-1}+\varphi(t)\right)\, dt\right) + \\ &\quad + \gamma_2 e^{\lambda_2 t_i}\int_{t_{i-1}+\tau}^{t_i} e^{-\lambda_2 t}\left(u_i + \varphi(t)\right)\, dt = e^{\lambda_2 T_i}\sigma_i + K_i, \end{aligned}$$

where

$$K_i = \gamma_2 e^{\lambda_2 t_i}\left(\int_{t_{i-1}}^{t_i} e^{-\lambda_2 t}\varphi(t)\, dt + \int_{t_{i-1}}^{t_{i-1}+\tau} e^{-\lambda_2 t} u_{i-1}\, dt + \int_{t_{i-1}+\tau}^{t_i} e^{-\lambda_2 t} u_i\, dt\right).$$

So,

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & e^{\lambda_2 T_4}\\ e^{\lambda_2 T_1} & 0 & 0 & 0\\ 0 & e^{\lambda_2 T_2} & 0 & 0\\ 0 & 0 & e^{\lambda_2 T_3} & 0\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix} + \begin{pmatrix}K_1\\ K_2\\ K_3\\ K_4\end{pmatrix}$$

and

$$\begin{aligned}
\sigma_1 &= \left(1 - e^{\lambda_2 T}\right)\left(K_2 e^{\lambda_2(T_2+T_3+T_4)} + K_3 e^{\lambda_2(T_3+T_4)} + K_4 e^{\lambda_2 T_4} + K_1\right) = l_0,\\
\sigma_2 &= \left(1 - e^{\lambda_2 T}\right)\left(K_3 e^{\lambda_2(T_1+T_3+T_4)} + K_4 e^{\lambda_2(T_1+T_4)} + K_1 e^{\lambda_2 T_1} + K_2\right) = -l,\\
\sigma_3 &= \left(1 - e^{\lambda_2 T}\right)\left(K_4 e^{\lambda_2(T_1+T_2+T_4)} + K_1 e^{\lambda_2(T_1+T_2)} + K_2 e^{\lambda_2 T_2} + K_3\right) = -l_0,\\
\sigma_4 &= \left(1 - e^{\lambda_2 T}\right)\left(K_1 e^{\lambda_2(T_1+T_2+T_3)} + K_2 e^{\lambda_2(T_2+T_3)} + K_3 e^{\lambda_3 T_3} + K_4\right) = l,
\end{aligned}$$

here $T = T_1 + T_2 + T_3 + T_4$ is a period of the solution (let it is multiple of T_φ). Consider the latest system as a system of linear equations with respect to γ_2, m (for example), i.e.

$$\sigma_1 = \Psi_1(m, \gamma_2) = l_0, \quad \sigma_2 = \Psi_2(m, \gamma_2) = -l, \quad \sigma_3 = \Psi_3(m, \gamma_2) = -l_0, \quad \sigma_4 = \Psi_4(m, \gamma_2) = l.$$

Suppose $\Psi_i \equiv -\Psi_{i+2}$ (it can be if the solution is origin-symmetric).
Denote

$$\begin{aligned}
\hat{\psi}_i(t) &= \sigma\left(t_i + t,\, t_i,\, \sigma_i,\, u_{i-1}\right), \quad t \in [0,\, \tau),\\
\psi_i(t) &= \sigma\left(t_i + \tau + t,\, t_i + \tau,\, \hat{\sigma}_1,\, u_i\right), \quad t \in [0,\, T_i - \tau)
\end{aligned}$$

Following result may be formulated.

Theorem 6. *Let the system*

$$\begin{cases}\Psi_1(m, \gamma_2) = l_0,\\ \Psi_2(m, \gamma_2) = -l.\end{cases}$$

has a solution such as for given $\gamma = \left(0,\, \gamma_2\right)'$ *and* m *conditions*

$$\begin{cases}
\hat{\psi}_1(t) > -l, & t \in [0,\, \tau),\\
\psi_1(t) > -l, & t \in [0,\, T_1 - \tau),\\
\hat{\psi}_2(t) > -l_0, & t \in [0,\, \tau),\\
\psi_2(t) > -l_0, & t \in [0,\, T_2 - \tau),\\
\hat{\psi}_3(t) < l, & t \in [0,\, \tau),\\
\psi_3(t) < l, & t \in [0,\, T_3 - \tau),\\
\hat{\psi}_4(t) > l_0, & t \in [0,\, \tau),\\
\psi_4(t) > l_0, & t \in [0,\, T_4 - \tau)
\end{cases} \tag{15}$$

are satis□ed. In that case system (14) has a stable T-periodic solution with switching points $\hat{s}_i$, *if* $\lambda_1 < 0$ *and*

$$T\, T_\varphi^{-1} \in \mathbb{N}.$$

Proof In order to prove the theorem it is enough to note that under above-listed conditions system (14) settles self-mapping of switching lines $\sigma = l_i$. Moreover, for any $x^{(i)}$ lying on switching line,

$$x_1^{(i+1)} = e^{\lambda_1 T} x_1^{(i)} + \Theta, \quad \Theta \in \mathbb{R},$$

and in general case ($\Theta \neq 0$) the latter difference equation has stable solution only if $\lambda_1 < 0$. ■
In order to pass onto variables z_i it is enough to effect linear transform (13).
Note that conditions (15) may be readily verified using mathematical symbolic packages.
Of course the statement Theorem 6 is just an outline. Further investigation of the system (11) requires specification of φ function, detailed computations are quite laborious.
On the analogy with the previous section a case of multiple delays can be observed.

7. Conclusion

The above results suppose further development. Investigation of stable modes of the forced system (10) is an individual complex task (systems with several delays may also be considered). Results similar to obtained in the last part can be outlined for periodic solutions of the system (10) having a quite complicated configuration (large amount of control switching point etc.).
Stabilization problem (i.e. how to choose setup variables of a system in order to put its steady state solution in a prescribed neighbourhood of the origin) was not discussed. This problem was elucidated in Zubov (1999), Zubov & Zubov (1996) for a bit different systems.

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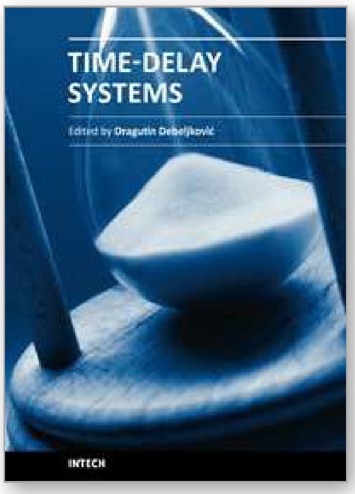

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Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, robotics, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of controllability, observability, robustness, optimization, adaptive control, pole placement and particularly stability and robustness stabilization for this class of systems, has been one of the main interests for many scientists and researchers during the last five decades.