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Modern Stochastic Thermodynamics

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For our beloved son Eugene

1. Introduction
Limitations of thermodynamics based on the quantum statistical mechanics

An increased interest in using equilibrium thermodynamics as an independent macrotheory can be observed in recent years. From a fundamental standpoint, thermodynamics gives an universal macrodescription of nature in which using specific micromodels of objects is unnecessary. From a pragmatic standpoint, there is obviously a demand for using thermodynamics both to describe the behavior of relatively small objects (nanoparticles, etc.) at low temperatures and to study high-energy physics (including the quark–gluon plasma).

As is well known, phenomenological thermodynamics is based on four laws. Among them, the zero law is basic. It relates the fundamental idea of thermal equilibrium of an object to its environment, called a *heat bath*. In this theory, in which all macroparameters are exactly defined, the zero law is a strict condition determining the concept of temperature:

$$T \equiv T_0, \tag{1}$$

where T is the object temperature and T_0 is the heat bath temperature.

In the same time there exists also statistical thermodynamics (ST). In its nonquantum version founders of which were Gibbs and Einstein {LaLi68},{Su05} all macroparameters are considered random values fluctuating about their means. It is assumed here that the concept of thermal equilibrium is preserved, but its content is generalized. It is now admitted that the object temperature experiences also fluctuations δT because of the thermal stochastic influence of the heat bath characterized by the Boltzmann constant k_{B}. As a result, the zero law takes the form of a soft condition, namely,

$$T = T_0 \pm \delta T = \langle T \rangle \pm \delta T. \tag{2}$$

Here the average object temperature $\langle T \rangle$ coincides with T_0 and $\langle (\delta T)^2 \rangle \equiv (\Delta T)^2$ has the meaning of the object temperature dispersion.

To preserve the thermodynamic character of this description, it is simultaneously assumed that the values of the dispersion of any macroparameter A_i is bounded by the condition $(\Delta A_i)^2 / \langle A_i \rangle^2 \leq 1$. This means that for the dispersion $(\Delta T)^2$ there is the requirement

$$\frac{(\Delta T)^2}{T_0^2} \leq 1. \tag{3}$$

In other words, the zero law of the nonquantum version of statistical thermodynamics is not just one condition (2) but the set of conditions (2) and (3).

We stress that nonquantum version of statistical thermodynamics (see chap. 12 in {LaLi68}) absolutely does not take the quantum stochastic influence characterized by the Planck constant $\hbar$ into account. At the same time, it is well known from quantum dynamics that the characteristics of an object can experience purely quantum fluctuations when there are no thermal effects. In the general case, both quantum and thermal types of environment stochastic influences determining macroparameters and their fluctuations are simultaneously observed in experiments. In this regard, it is necessary to develop a theory such that the approaches of quantum mechanics and nonquantum version of statistical thermodynamics can be combined.

Today, there exists a sufficiently widespread opinion that thermodynamics based on quantum statistical mechanics (QSM-based thermodynamics) has long played the role of such a theory quite effectively. But this theoretical model is probably inadequate for solving a number of new problems. In our opinion, this is due to the following significant factors.

First, QSM-based thermodynamics is not a consistent quantum theory because it plays the role of a quasiclassical approximation in which the nonzero energy of the ground state is not taken into account. Second, the theory is not a consistent statistical theory because it does not initially contain fluctuations of intensive macroparameters (primarily, of temperature). However, the temperature fluctuations in low-temperature experiments are sufficiently noticeable for small objects, including nanoparticles and also for critical phenomena. Third, the assertion that the minimal entropy is zero in it, is currently very doubtful. Fourth, in this theory, the expression $\Theta = k_B T$ is used as a modulus of the distribution for any objects at any temperature. This corresponds to choosing the classical model of the heat bath {Bog67} as a set of weakly coupled classical oscillators. Then a microobject with quantized energy is placed in it. Thus, quantum and thermal influences are considered as additive. Fifth, in this theory at enough low temperatures the condition (3) is invalid for relative fluctuations of temperature. As a result, in QSM-based thermodynamics, it is possible to calculate the means of the majority of extensive macroparameters with the account of quantum stochastic influence. However, using the corresponding apparatus to calculate fluctuations of the same macroparameters leads to the violation of condition that is analogical one (3). This means that full value statistical thermodynamics as a macrotheory cannot be based completely on QSM as a microtheory.

To obtain a consistent quantum-thermal description of natural objects, or modern stochastic thermodynamics (MST), in our opinion, it is possible to use two approaches. Nevertheless, they are both based on one general idea, namely, replacing the classical model of the heat bath with an adequate quantum model, or a *quantum heat bath* (QHB) {Su99}.

The first of these approaches is described in the Sect. 1 {Su08}. We modify the macrodescription of objects in the heat bath by taking quantum effects into account in the framework of nonquantum version of statistical thermodynamics with an inclusion of temperature fluctuations but without using the operator formalism. In this case, based on intuitive considerations, we obtain a theory of effective macroparameters (TEM) as a macrotheory.

In the Sect. 2 we modify standard quantum mechanics taking thermal effects into account {SuGo09}. As a result, we formulate a quantum-thermal dynamics or, briefly, $(\hbar, k)$-dynamics ($\hbar k D$) as a microtheory. The principal distinction from QSM is that in such a theory, the state of a microobject under the conditions of contact with the QHB is generally described not by

the density matrix but by a temperature-dependent complex wave function.

In the Sect. 3 we overcome the main paradox appearing in QSM-based thermodynamics at calculation of macroparameters fluctuations. It is that at account of quantum effects its results fall outside the scope of the thermodynamics. We develop the theory of the effective macroparameters fluctuations (TEMF) combining TEM and $\hbar k D$. We also investigate effective macroparameters obeying the uncertainties relations (URs) and offer a criterion for the choice of conjugate quantities.

2. Theory of effective macroparameters as a macroscopic ground of modern stochastic thermodynamics

At first we construct MST in the form of a macrotheory or TEM. That is a generalization of nonquantum version of statistical thermodynamics. The development of this theory is based on a main MST postulate reduced to statements:

A. Stochastic influences of quantum and thermal types are realized by an environment to which the QHB model is assigned.

B. The state of thermal equilibrium between the object and the QHB is described by an effective temperature.

C. The physical characteristics of objects of any complexity at any temperature are described by effective macroparameters to which random c-number quantities are assigned.

D. The main thermodynamic relations are formulated for the corresponding effective macroparameters; moreover, their standard forms are preserved, including zero law (2)-(3).

2.1 Effective temperature

We note that by changing the form of the zero law from (1) to (2) - (3), we take into account that the object temperature can fluctuate. Therefore, the only possibility (probably still remaining) is to modify the model of the heat bath, which is a source of stochastic influences, by organically including a quantum-type influence in it.

Because an explicit attempt to modify the heat bath model is made by as for the first time, it is useful first to make clear what is tacitly taken for such a model in the nonquantum version of statistical thermodynamics. As follows from Chap. 9 in the Gibbs's monograph {Gi60}, it is based on the canonical distribution

$$dw(\mathcal{E}) = e^{(F-\mathcal{E})/\Theta} d\mathcal{E} \tag{4}$$

in the macroparameters space [1]. The object energy $\mathcal{E} = \mathcal{E}(V, T)$ in it is a random quantity whose fluctuations (for $V = $ const) depend on object temperature fluctuations according to zero law (2)-(3); F is the free energy determined by the normalization condition. The distribution modulus

$$\Theta \equiv k_{\mathrm{B}} T_0 \tag{5}$$

has a sense of the energy typical of a definite heat bath model.

Up to now, according to the ideas of Bogoliubov {Bog67}, a heat bath is customarily modeled by an infinite set of normal modes each of which can be treated as an excitation of a chain

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[1] We emphasize that distribution (4) is similar to the canonical distribution in classical statistical mechanics (CSM) only in appearance. The energy $\varepsilon = \varepsilon(p, q)$ in the latter distribution is also a random quantity, but its fluctuations depend on the fluctuations of the microparameters p and q at the object temperature defined by the formula (1).

of weakly coupled oscillators. As follows from experiments, the quantity $k_B T_0$ in relatively narrow ranges of frequencies and temperatures has the meaning of the average energy $\langle \varepsilon_{cl} \rangle$ of the classical normal mode. It can therefore be concluded from formulae (4) and (5) that the heat bath model that can be naturally called classical is used in the nonquantum version of statistical thermodynamics. From a modern standpoint, the experimental data in some cases cannot be interpreted using such a model, on which, we stress, QSM is also based.

In what follows, we propose an alternative method for simultaneously including quantum- and thermal-type stochastic influences. According to the main MST postulate, we pass from the classical heat bath model to a more general quantum model, or QHB. As a result, all effects related to both types of environment stochastic influences on the objects can be attributed to the generalized heat bath. However, the thermodynamic language used to describe thermal equilibrium can be preserved, i.e., we can explicitly use no the operator formalism in this language. For this, as the QHB model, we propose to choose the set consisting of an infinite number of quantum normal modes, each with the average energy

$$\langle \varepsilon_{qu} \rangle = \frac{\hbar \omega}{2} + \hbar \omega \left(e^{\hbar \omega / (k_B T_0)} - 1 \right)^{-1} = \frac{\hbar \omega}{2} \coth \frac{\hbar \omega}{2 k_B T_0} \tag{6}$$

over the entire ranges of frequencies and temperatures, which agrees with experiments. This means that in the QHB, we determine the expression for the distribution modulus Θ by the more general condition $\Theta = \langle \varepsilon_{qu} \rangle$, instead of the condition $\Theta = \langle \varepsilon_{cl} \rangle$ typical of the classical model.

Further, according to the main MST postulate, we propose to write the quantity Θ as

$$\Theta \equiv k_B (T_{ef})_0. \tag{7}$$

It is significant that the introduced quantity

$$(T_{ef})_0 \equiv \frac{\langle \varepsilon_{qu} \rangle}{k_B} = \frac{\hbar \omega}{2 k_B} \coth \frac{\hbar \omega}{2 k_B T_0} \tag{8}$$

has the meaning of the effective QHB temperature. It fixes the thermal equilibrium condition in the case when stochastic influences of both types are taken into account on equal terms. It depends on both fundamental constants $\hbar$ and k_B.

We could now formulate a zero law similar to (2)-(3) as the interrelation condition for the effective object and QHB temperatures T_{ef} and $(T_{ef})_0$. But we restrict ourself here to the consideration of problems in which it is not necessary to take the fluctuations of the effective object temperature into account. We therefore set

$$(T_{ef})_0 \equiv T_{ef} \quad \text{and} \quad T_0 \equiv T$$

in all formulae of Sections 1 and 2. The quantum generalization of macroparameters fluctuations theory (TEMF) is the subject of the Sect 3.

We call attention to the fact that the effective object temperature T_{ef} is a function of two object characteristics ω and T. In this case, equilibrium thermal radiation with a continuous spectrum is manifested as a QHB with a temperature T on the Kelvin scale. Under these conditions, we have not only $T = T_0$ but also $\omega = \omega_0$ as the thermal equilibrium state is reached, i.e., it is as if the object made a resonance choice of one of the QHB modes whose frequency ω_0 coincides with its characteristic frequency ω. It is necessary to choose the corresponding frequency from either the experiment or some intuitive considerations. In

this regard, we can assume that the MST can at least be applied to a wide class of objects whose periodic or conditionally periodic motions can be assigned to independent degrees of freedom.

We note that the frequency ω and the temperature T (and consequently the effective temperature T_{ef}) are intensive quantities, which stresses that they are conceptually close to each other but makes them qualitatively different from extensive quantities (for example, energy), for which additivity is typical. They are holistic characteristics of the state of the object–environment system and have the transitivity property. It is significant that the characteristics ω, T, and T_{ef}, being c-number quantities, are not initially related to the number of observables to which operators are assigned in quantum theory.

We can elucidate the physical meaning of the effective temperature T_{ef} by considering its behavior in the limiting cases. Thus, as the temperature (on the Kelvin scale) $T \rightarrow 0$, the effective temperature T_{ef} becomes nonzero,

$$T_{ef} \rightarrow \frac{\hbar\omega}{2k_{\mathrm{B}}} \equiv T_{ef}^0, \tag{9}$$

where T_{ef}^0 has the meaning of the minimal effective temperature of the object with the characteristic frequency ω.

The effective temperature in turn becomes

$$T_{ef} \equiv T_{ef}^0 \coth \frac{T_{ef}^0}{T} \rightarrow T\left[1 + \frac{1}{3}\left(\frac{T_{ef}^0}{T}\right)^2 + \cdots\right] \tag{10}$$

in the limit of high temperatures T. Of course, the concepts of low and high temperatures for each object essentially depend on the ratio T_{ef}^0/T.

2.2 Effective entropy

To calculate the effective macroparameters in terms of the corresponding distribution function, we must generalize canonical distribution (4) introduced by Gibbs in the nonquantum version of statistical thermodynamics. As above, according to the main MST postulate, this generalization reduces to replacing expression (5) for the distribution modulus Θ with expression (7), i.e., to replacing $T = T_0$ with $T_{ef} = (T_{ef})_0$. The desired distribution thus becomes

$$dw(\mathcal{E}) = \rho(\mathcal{E})\,d\mathcal{E} = \frac{1}{k_{\mathrm{B}}T_{ef}} e^{-\mathcal{E}/(k_{\mathrm{B}}T_{ef})}\,d\mathcal{E}, \tag{11}$$

where $\mathcal{E}$ is the random energy of the object's independent degree of freedom to which the model of the oscillator with the frequency ω is assigned.

Based on distribution (11), we can calculate the internal energy of the object as a macroparameter:

$$\mathcal{E}_{ef} = \int \mathcal{E}\rho(\mathcal{E})\,d\mathcal{E} = k_{\mathrm{B}}T_{ef}. \tag{12}$$

Because $\mathcal{E}_{ef}$ with account (8) coincides with $\langle \varepsilon_{\mathrm{qu}} \rangle$ of form (6), this means that the quantum oscillator in the heat bath is chosen as an object model.

To calculate the effective entropy S_{ef} of such an object, it is convenient to write formula (11) in the form in which the distribution density $\tilde{\rho}(\mathcal{E}) = \rho(\mathcal{E})\hbar\omega/2$ is dimensionless,

$$dw(\mathcal{E}) = \tilde{\rho}(\mathcal{E})\left(\frac{\hbar\omega}{2}\right)^{-1}d\mathcal{E} = e^{(F_{ef}-\mathcal{E})/(k_{\mathrm{B}}T_{ef})}\left(\frac{\hbar\omega}{2}\right)^{-1}d\mathcal{E}, \tag{13}$$

where the effective free energy is given by

$$F_{ef} = -k_{\mathrm{B}}T_{ef}\log\left(\frac{T_{ef}}{T_{ef}^0}\right). \tag{14}$$

We then obtain

$$S_{ef} = -k_{\mathrm{B}}\int\tilde{\rho}(\mathcal{E})\log\tilde{\rho}(\mathcal{E})\left(\frac{\hbar\omega}{2}\right)^{-1}d\mathcal{E} =$$

$$= k_{\mathrm{B}}\left[1+\log\left(\coth\frac{\hbar\omega}{2k_{\mathrm{B}}T}\right)\right] = k_{\mathrm{B}}\left[1+\log\left(\coth\frac{T_{ef}^0}{T}\right)\right]. \tag{15}$$

It follows from formula (15) that in the high-temperature limit $T \gg T_{ef}^0$, the effective entropy is written as

$$S_{ef} \to k_{\mathrm{B}}\log T + \mathrm{const}, \tag{16}$$

which coincides with the expression for the oscillator entropy in thermodynamics based on classical statistical mechanics (CSM-based thermodynamics). At the same time, in the low-temperature limit $T \ll T_{ef}^0$, the effective entropy is determined by the world constant k_{B}:

$$S_{ef} \to S_{ef}^0 = k_{\mathrm{B}}. \tag{17}$$

Thus, in TEM, the behavior of the effective entropy of the degree of freedom of the object for which the periodic motion is typical corresponds to the initial formulation of Nernst's theorem, in which the minimum entropy is nonzero. Moreover, the range of temperatures T where we have $S_{ef} \approx S_{ef}^0$ can be very considerable, depending on the ratio T_{ef}^0/T.

It is obvious that using the model of the QHB, we can combine the quantum- and thermal-type influences (traditionally considered as specific influences only for the respective micro- and macrolevels) to form a holistic stochastic influence in the TEM framework. But using such an approach, we need not restrict ourself to generalizing only the traditional macroparameters, such as temperature and entropy. It becomes possible to give a meaning to the concept of effective action, as a new macroparameter which is significantly related to the quantum-type stochastic influence on the microlevel.

2.3 Effective action as a new macroparameter

The problem of introducing the concept of action into thermodynamics and of establishing the interrelation between the two widespread (but used in different areas of physics) quantities (entropy and action) has attracted the attention of many the most outstanding physicists, including Boltzmann {Bol22}, Boguslavskii, de Broglie. But the results obtained up to now were mainly related to CSM-based thermodynamics, and quantum effects were taken into account only in the quasiclassical approximation. Our aim is to extend them to the TEM. To do this, we choose the harmonic oscillator as an initial model of a periodically moving object. If we pass from the variables p and q to the action–angle variables when analyzing it in the framework of classical mechanics, then we can express the action j (having the meaning of a generalized momentum) in terms of the oscillator energy ε as

$$j = \frac{\varepsilon}{\omega}. \tag{18}$$

In passing to thermodynamics, we should preserve interrelation (18) between the action and the energy. In the same time, in place of the microparameters j and ε we use the corresponding macroparameters in this case, namely, the average quantities $\langle j \rangle$ and $\langle \varepsilon \rangle$. Their specific expressions depend on the choice of the heat bath model (classical or non-classical) used for averaging.

It is quite natural that in the framework of CSM-based thermodynamics, Boltzmann assumed that

$$\mathcal{J}_T \equiv \langle j \rangle = \frac{\langle \varepsilon_{\mathrm{cl}} \rangle}{\omega} = \frac{k_{\mathrm{B}} T}{\omega}. \tag{19}$$

Following the same idea, we determine the effective action $\mathcal{J}_{ef}$ in the TEM framework by the formula

$$\mathcal{J}_{ef} = \langle j \rangle = \frac{\langle \varepsilon_{\mathrm{qu}} \rangle}{\omega} = \frac{\mathcal{E}_{ef}}{\omega} = \frac{k_{\mathrm{B}} T_{ef}}{\omega}. \tag{20}$$

This means that in the TEM, we start from the fact that the effective action for all objects to which the model of the quantum oscillator in the QHB is applicable has the form

$$\mathcal{J}_{ef} = \frac{\hbar}{2} \coth \frac{\hbar \omega}{2 k_{\mathrm{B}} T} = \mathcal{J}_{ef}^0 \coth \frac{T_{ef}^0}{T}, \tag{21}$$

where accordingly (20)

$$\mathcal{J}_{ef}^0 = \frac{k_B T_{ef}^0}{\omega} = \frac{\hbar}{2} \tag{22}$$

is the minimal effective action for $T \to 0$. Of course, in the limit $T \gg T_{ef}^0$, we have the effective action

$$\mathcal{J}_{ef} \to \mathcal{J}_T \left[1 + \frac{1}{3} \left(\frac{T_{ef}^0}{T} \right)^2 + \cdots \right],$$

i.e., it goes to expression (19) obtained in the CSM-based thermodynamics. Thus, both at low and high temperatures the formulae (22) and (19) for the effective action $\mathcal{J}_{ef}$ are written by the minimal effective temperature T_{ef}^0. This means that even purely quantum influence (at $T = 0$) can be interpreted as a peculiar thermal influence. Thus, one cannot assume that quantum and thermal influences can be considered separately. In other words, they are non-additive notions.

As is well-known, the original Planck formula for the average energy of the quantum oscillator in QSM

$$\mathcal{E}_{\mathrm{quasi}} = \frac{\hbar \omega}{e^{\hbar \omega / (k_{\mathrm{B}} T)} - 1} \equiv \mathcal{E}_{ef} - \frac{\hbar \omega}{2} \tag{23}$$

is only applicable in the quasiclassical approximation framework. Substituting the expression $\mathcal{E}_{\mathrm{quasi}}$ of form (23) in formula (20) instead of $\mathcal{E}_{ef}$, we also obtain the effective action in the quasiclassical approximation:

$$\mathcal{J}_{\mathrm{quasi}} = \frac{\mathcal{E}_{\mathrm{quasi}}}{\omega} = \frac{\hbar}{e^{\hbar \omega / (k_{\mathrm{B}} T)} - 1} \equiv \mathcal{J}_{ef} - \frac{\hbar}{2}. \tag{24}$$

The quasiclassical nature of expressions (23) and (24) is manifested, in particular, in the fact that these both quantities tend to zero as $T \to 0$.

2.4 The interrelation between the effective action and the effective entropy

To establish the interrelation between the action and the entropy, Boltzmann assumed that the isocyclic motions of the oscillator in mechanics for which $\omega = \text{const}$ correspond to the isothermal processes in thermodynamics. In this case, the oscillator energy can be changed under external influence that can be treated as the work δA^{dis} of dissipative forces equivalent to the heat δQ.

Generalizing this idea, we assume that every energy transferred at stochastic influence (quantum and thermal) in the TEM can be treated as the effective work $\delta A_{ef}^{\text{dis}}$ of dissipative forces equivalent to the effective heat δQ_{ef}. This means that for isothermal processes, the same change in the effective energy $d\mathcal{E}_{ef}$ of the macroobject to which the model of the quantum oscillator in the QHB can be assigned can be represented in two forms

$$d\mathcal{E}_{ef} = \delta A_{ef}^{\text{dis}} = \omega \, d\mathcal{J}_{ef} \quad \text{or} \quad d\mathcal{E}_{ef} = \delta Q_{ef} = T_{ef} \, dS_{ef}. \tag{25}$$

Furthermore, following Boltzmann, we choose the ratio $d\mathcal{E}_{ef}/\mathcal{E}_{ef}$ as a measure of energy transfer from the QHB to the object in such processes. The numerator and denominator in this ratio can be expressed in terms of either the effective action $\mathcal{J}_{ef}$ or the effective entropy S_{ef} and effective temperature T_{ef} using formulas (25) and (20). Equating the obtained expressions for the ratio $d\mathcal{E}_{ef}/\mathcal{E}_{ef}$, we obtain the differential equation

$$\frac{d\mathcal{E}_{ef}}{\mathcal{E}_{ef}} = \frac{\omega \, d\mathcal{J}_{ef}}{\omega \mathcal{J}_{ef}} = \frac{T_{ef} \, dS_{ef}}{k_{\text{B}} T_{ef}} \tag{26}$$

relating the effective entropy to the effective action. Its solution has the form

$$S_{ef} = k_{\text{B}} \int \frac{d\mathcal{J}_{ef}}{\mathcal{J}_{ef}} = k_{\text{B}} \log \frac{\mathcal{J}_{ef}}{J_0} = k_{\text{B}} \log \left(\frac{\hbar}{2J_0} \coth \frac{\hbar\omega}{2k_{\text{B}}T} \right), \tag{27}$$

where J_0 is the arbitrary constant of action dimensionality.

Choosing the quantity $\hbar/2e$ as J_0, where e is the base of the natural logarithms, we can make expression (27) coincides with the expression for the effective entropy S_{ef} of form (15). Taking into account that $S_{ef}^0 = k_{\text{B}}$, we have

$$S_{ef} = S_{ef}^0 \left[1 + \log \frac{\mathcal{J}_{ef}}{\mathcal{J}_{ef}^0} \right] = S_{ef}^0 \left[1 + \log \coth \frac{T_{ef}^0}{T} \right]. \tag{28}$$

For the entropy of the quantum oscillator in QSM-based thermodynamics, i.e. in the quasiclassical approximation, the well-known expression

$$S_{\text{quasi}} = -k_{\text{B}} \left\{ \frac{\hbar\omega}{k_{\text{B}}T} \left(1 - e^{\hbar\omega/(k_{\text{B}}T)} \right)^{-1} + \log \left(1 - e^{-\hbar\omega/(k_{\text{B}}T)} \right) \right\} \tag{29}$$

is applicable. It will be interesting to compare (28) with the analogical expression from QSM-based thermodynamics. For this goal we rewrite the formula (29), taking into account (24) in the form

$$S_{\text{quasi}} = \frac{\omega}{T} \mathcal{J}_{\text{quasi}} + k_B \log(1 + \frac{\mathcal{J}_{\text{quasi}}}{\hbar}). \tag{30}$$

In contrast to S_{ef} in the form (28) the quantity S_{quasi} tends to zero as $T \to 0$.

2.5 The first holistic stochastic-action constant

We note that according to formulas (21) and (28), the ratio of the effective action to the effective entropy is given by

$$\frac{\mathcal{J}_{ef}}{S_{ef}} = \frac{\mathcal{J}_{ef}^0}{S_{ef}^0} \cdot \frac{\coth(T_{ef}^0/T)}{1 + \log\coth(T_{ef}^0/T)} = \varkappa \frac{\coth(\varkappa\omega/T)}{1 + \log\coth(\varkappa\omega/T)}. \tag{31}$$

In this expression,

$$\varkappa \equiv \frac{\mathcal{J}_{ef}^0}{S_{ef}^0} = \frac{\hbar}{2k_B} \tag{32}$$

is the minimal ratio (31) for $T \ll T_{ef}^0$.

In our opinion, the quantity

$$\varkappa = 3.82 \cdot 10^{-12}\,\mathrm{K} \cdot s \tag{33}$$

is not only the notation for one of the possible combinations of the world constants $\hbar$ and k_B. It also has its intrinsic physical meaning. In addition to the fact that the ratio $\mathcal{J}_{ef}/S_{ef}$ of form (31) at any temperature can be expressed in terms of this quantity, it is contained in definition (2.5) of the effective temperature

$$T_{ef} = \varkappa\omega \coth \frac{\varkappa\omega}{T} \tag{34}$$

and also in the Wien's displacement law $T/\omega_{\max} = 0.7\varkappa$ for equilibrium thermal radiation. Starting from the preceding, we can formulate the hypothesis according to which the quantity $\varkappa$ plays the role of the *first* constant essentially characterizing the holistic stochastic action of environment on the object.

Hence, the minimal ratio of the action to the entropy in QSM-based thermodynamics is reached as $T \to 0$ and is determined by the formula

$$\frac{\mathcal{J}_{quasi}}{S_{quasi}} = \frac{T}{\omega}\left[1 + \frac{k_B T}{\omega \mathcal{J}_{quasi}}\log\left(1 + \frac{\mathcal{J}_{quasi}}{\hbar}\right)\right]^{-1} \to \frac{T}{\omega}\left[1 + \frac{k_B T}{\hbar\omega}\right]^{-1} \to 0. \tag{35}$$

We have thus shown that not only $\mathcal{J}_{quasi} \to 0$ and $S_{quasi} \to 0$ but the ratio $\mathcal{J}_{quasi}/S_{quasi} \to 0$ in this microtheory too. This result differs sharply from the limit $\mathcal{J}_{ef}/S_{ef} \to \varkappa \neq 0$ for the corresponding effective quantities in the TEM. Therefore, it is now possible to compare the two theories (TEM and QSM) experimentally by measuring the limiting value of this ratio.

The main ideas on which the QST as a macrotheory is based were presented in the foregoing. The stochastic influences of quantum and thermal types over the entire temperature range are taken into account simultaneously and on equal terms in this theory. As a result, the main macroparameters of this theory are expressed in terms of the single macroparameter $\mathcal{J}_{ef}$ and combined fundamental constant $\varkappa = \hbar/2k_B$. The experimental detection $\varkappa$ as the minimal nonzero ratio $\mathcal{J}_{ef}/S_{ef}$ can confirm that the TEM is valid in the range of sufficiently low temperatures. The first indications that the quantity $\varkappa$ plays an important role were probably obtained else in Andronikashvili's experiments (1948) on the viscosity of liquid helium below the λ point.

3. $(\hbar, k)$-dynamics as a microscopic ground of modern stochastic thermodynamics

In this section, following ideas of paper {Su06}, where we introduced the original notions of $\hbar k$D, we develop this theory further as a microdescription of an object under thermal equilibrium conditions {SuGo09}. We construct a model of the object environment, namely, QHB at zero and finite temperatures. We introduce a new microparameter, namely, the stochastic action operator, or Schrödingerian. On this ground we introduce the corresponding macroparameter, the effective action, and establish that the most important effective macroparameters —internal energy, temperature, and entropy—are expressed in terms of this macroparameter. They have the physical meaning of the standard macroparameters for a macrodescription in the frame of TEM describing in the Sect.1.

3.1 The model of the quantum heat bath: the "cold" vacuum

In constructing the $\hbar k$D, we proceed from the fact that no objects are isolated in nature. In other words, we follow the Feynman idea, according to which any system can be represented as a set of the object under study and its environment (the "rest of the Universe"). The environment can exert both regular and stochastic influences on the object. Here, we study only the stochastic influence. Two types of influence, namely, quantum and thermal influences characterized by the respective Planck and Boltzmann constants, can be assigned to it.

To describe the environment with the holistic stochastic influence we introduce a concrete model of environment, the QHB. It is a natural generalization of the classical thermal bath model used in the standard theories of thermal phenomena {Bog67}, {LaLi68}. According to this, the QHB is a set of weakly coupled quantum oscillators with all possible frequencies. The equilibrium thermal radiation can serve as a preimage of such a model in nature.

The specific feature of our understanding of this model is that we assume that we must apply it to both the "thermal" ($T \neq 0$) and the "cold" ($T = 0$) vacua. Thus, in the sense of Einstein, we proceed from a more general understanding of the thermal equilibrium, which can, in principle, be established for any type of environmental stochastic influence (purely quantum, quantum-thermal, and purely thermal).

We begin our presentation by studying the "cold" vacuum and discussing the description of a single quantum oscillator from the number of oscillators forming the QHB model for $T = 0$ from a new standpoint. For the purpose of the subsequent generalization to the case $T \neq 0$, not its well-known eigenstates $\Psi_n(q)$ in the q representation but the coherent states (CS) turn out to be most suitable.

But we recall that the lowest state in the sets of both types is the same. In the occupation number representation, the "cold" vacuum in which the number of particles is $n = 0$ corresponds to this state. In the q representation, the same ground state of the quantum oscillator is in turn described by the real wave function

$$\Psi_0(q) = [2\pi(\Delta q_0)^2]^{-1/4} e^{-q^2/4(\Delta q_0)^2}. \tag{36}$$

In view of the properties of the Gauss distribution, the Fourier transform $\Psi_0(p)$ of this function has a similar form (with q replaced with p); in this case, the respective momentum and coordinate dispersions are

$$(\Delta p_0)^2 = \frac{\hbar m \omega}{2}, \qquad (\Delta q_0)^2 = \frac{\hbar}{2m\omega}. \tag{37}$$

As is well known, CS are the eigenstates of the non-Hermitian particle annihilation operator $\hat{a}$ with complex eigenvalues. But they include one isolated state $|0_a\rangle$ of the particle vacuum in

which the eigenvalue of $\hat{a}$ is zero

$$\hat{a}|0_a\rangle = 0|0_a\rangle, \quad \text{or} \quad \hat{a}\Psi_0(q) = 0. \tag{38}$$

In what follows, it is convenient to describe the QHB in the q representation. Therefore, we express the annihilation operator $\hat{a}$ and the creation operator $\hat{a}^\dagger$ in terms of the operators $\hat{p}$ and $\hat{q}$ using the traditional method. We have

$$\hat{a} = \frac{1}{2}\left(\frac{\hat{p}}{\sqrt{\Delta p_0^2}} - i\frac{\hat{q}}{\sqrt{\Delta q_0^2}}\right), \quad \hat{a}^\dagger = \frac{1}{2}\left(\frac{\hat{p}}{\sqrt{\Delta p_0^2}} + i\frac{\hat{q}}{\sqrt{\Delta q_0^2}}\right). \tag{39}$$

The particle number operator then becomes

$$\hat{N}_a = \hat{a}^\dagger\hat{a} = \frac{1}{\hbar\omega}\left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2} - \frac{\hbar\omega}{2}\hat{I}\right), \tag{40}$$

where $\hat{I}$ is the unit operator. The sum of the first two terms in the parentheses forms the Hamiltonian $\hat{\mathcal{H}}$ of the quantum oscillator, and after multiplying relations (40) by $\hbar\omega$ on the left and on the right, we obtain the standard interrelation between the expressions for the Hamiltonian in the q and n representations:

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2} = \hbar\omega\left(\hat{N}_a + \frac{1}{2}\hat{I}\right). \tag{41}$$

From the thermodynamics standpoint, we are concerned with the effective internal energy of the quantum oscillator in equilibrium with the "cold" QHB. Its value is equal to the mean of the Hamiltonian calculated over the state $|0_a\rangle \equiv |\Psi_0(q)\rangle$:

$$\mathcal{E}_{ef}^0 = \langle\Psi_0(q)|\hat{\mathcal{H}}|\Psi_0(q)\rangle = \hbar\omega\langle\Psi_0(q)|\hat{N}_a|\Psi_0(q)\rangle + \frac{\hbar\omega}{2} = \frac{\hbar\omega}{2} = \varepsilon_0. \tag{42}$$

It follows from formula (42) that in the given case, the state without particles coincides with the state of the Hamiltonian with the minimal energy ε_0. The quantity ε_0, traditionally treated as the zero point energy, takes the physical meaning of a macroparameter, or the effective internal energy $\mathcal{E}_{ef}^0$ of the quantum oscillator in equilibrium with the "cold" vacuum.

3.2 The model of the quantum heat bath: passage to the "thermal" vacuum

We can pass from the "cold" to the "thermal" vacuum using the Bogoliubov (u,v) transformation with the complex temperature-dependent coefficients { SuGo09}

$$u = \left(\frac{1}{2}\coth\frac{\hbar\omega}{2k_BT} + \frac{1}{2}\right)^{1/2}e^{i\pi/4}, \quad v = \left(\frac{1}{2}\coth\frac{\hbar\omega}{2k_BT} - \frac{1}{2}\right)^{1/2}e^{-i\pi/4}. \tag{43}$$

In the given case, this transformation is canonical but leads to a unitarily nonequivalent representation because the QHB at any temperature is a system with an infinitely large number of degrees of freedom.

In the end, such a transformation reduces to passing from the set of quantum oscillator CS to a more general set of states called the thermal correlated CS (TCCS) {Su06}. They are selected because they ensure that the Schrödinger coordinate–momentum uncertainties relation is saturated at any temperature.

From the of the second-quantization apparatus standpoint, the Bogoliubov (u, v) transformation ensures the passage from the original system of particles with the "cold" vacuum $|0_a\rangle$ to the system of quasiparticles described by the annihilation operator $\hat{b}$ and the creation operator $\hat{b}^{\dagger}$ with the "thermal" vacuum $|0_b\rangle$. In this case, the choice of transformation coefficients (43) is fixed by the requirement that for any method of description, the expression for the mean energy of the quantum oscillator in thermal equilibrium be defined by the Planck formula (6)

$$\mathcal{E}_{Pl} = \left[\langle \Psi_T(q) | \hat{\mathcal{H}} | \Psi_T(q) \rangle \right] = \langle \varepsilon_{qu} \rangle = \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2k_B T}, \tag{44}$$

which can be obtained from experiments. As shown in {Su06}, the state of the "thermal" vacuum $|0_b\rangle \equiv |\Psi_T(q)\rangle$ in the q representation corresponds to the complex wave function

$$\Psi_T(q) = [2\pi(\Delta q_{ef})^2]^{-1/4} \exp\left\{ -\frac{q^2}{4(\Delta q_{ef})^2}(1 - i\alpha) \right\}, \tag{45}$$

where

$$(\Delta q_{ef})^2 = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2k_B T}, \qquad \alpha = \left[\sinh \frac{\hbar\omega}{2k_B T} \right]^{-1}. \tag{46}$$

For its Fourier transform $\Psi_T(p)$, a similar expression with the same coefficient α and

$$(\Delta p_{ef})^2 = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2k_B T} \tag{47}$$

holds. We note that the expressions for the probability densities $\rho_T(q)$ and $\rho_T(p)$ have already been obtained by Bloch (1932), but the expressions for the phases that depend on the parameter α play a very significant role and were not previously known. It is also easy to see that as $T \to 0$, the parameter $\alpha \to 0$ and the function $\Psi_T(q)$ from TCCS passes to the function $\Psi_0(q)$ from CS.

Of course, the states from TCCS are the eigenstates of the non-Hermitian quasiparticle annihilation operator $\hat{b}$ with complex eigenvalues. They also include one isolated state of the quasiparticle vacuum in which the eigenvalue of b is zero,

$$\hat{b}|0_b\rangle = 0|0_b\rangle, \quad \text{or} \quad \hat{b}\Psi_T(q) = 0. \tag{48}$$

Using condition (48) and expression (45) for the wave function of the "thermal" vacuum, we obtain the expression for the operator $\hat{b}$ in the q representation:

$$\hat{b} = \frac{1}{2}\left(\coth \frac{\hbar\omega}{2k_B T} \right)^{\frac{1}{2}} \left[\frac{\hat{p}}{\sqrt{\Delta p_0^2}} - i\frac{\hat{q}}{\sqrt{\Delta q_0^2}}\left(\coth \frac{\hbar\omega}{2k_B T} \right)^{-1}(1 - i\alpha) \right]. \tag{49}$$

The corresponding quasiparticle creation operator has the form

$$\hat{b}^{\dagger} = \frac{1}{2}\left(\coth \frac{\hbar\omega}{2k_B T} \right)^{\frac{1}{2}} \left[\frac{\hat{p}}{\sqrt{\Delta p_0^2}} + i\frac{\hat{q}}{\sqrt{\Delta q_0^2}}\left(\coth \frac{\hbar\omega}{2k_B T} \right)^{-1}(1 + i\alpha) \right]. \tag{50}$$

We can verify that as $T \to 0$, the operators $\hat{b}^{\dagger}$ and $\hat{b}$ for quasiparticles pass to the operators $\hat{a}^{\dagger}$ and $\hat{a}$ for particles.

Acting just as above, we obtain the expression for the effective Hamiltonian, which is proportional to the quasiparticle number operator in the q representation

$$\hat{\mathcal{H}}_{ef} = \hbar\omega\hat{N}_b = \coth\frac{\hbar\omega}{2k_BT}\left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2}\right) - \frac{\hbar\omega}{2}\left(\hat{I} + \frac{\alpha}{\hbar}\{\hat{p},\hat{q}\}\right), \tag{51}$$

where we take $1 + \alpha^2 = \coth^2(\hbar\omega/2k_BT)$ into account. Obviously, $\hat{\mathcal{H}}_{ef}\Psi_T(q) = 0$, i.e. $\Psi_T(q)$-an eigenfunction of $\hat{\mathcal{H}}_{ef}$.

Passing to the original Hamiltonian, we obtain

$$\hat{\mathcal{H}} = \hbar\omega\left(\coth\frac{\hbar\omega}{2k_BT}\right)^{-1}\left[\hat{N}_b + \frac{1}{2}\left(\hat{I} + \frac{\alpha}{\hbar}\{\hat{p},\hat{q}\}\right)\right]. \tag{52}$$

We stress that the operator $\{\hat{p},\hat{q}\}$ in formula (52) can also be expressed in terms of bilinear combinations of the operators $\hat{b}^\dagger$ and $\hat{b}$, but they differ from the quasiparticle number operator. This means that the operators $\hat{\mathcal{H}}$ and $\hat{N}_b$ do not commute and that the wave function of form (45) characterizing the state of the "thermal" vacuum is therefore not the eigenfunction of the Hamiltonian $\hat{\mathcal{H}}$.

As before, we are interested in the macroparameter, namely, the effective internal energy $\mathcal{E}_{ef}$ of the quantum oscillator now in thermal equilibrium with the "thermal" QHB. Calculating it just as in Sec. 3.1, we obtain

$$\mathcal{E}_{ef} = \hbar\omega\left[\langle\Psi_T(q)|\hat{N}_b|\Psi_T(q)\rangle\right] + \frac{\hbar\omega}{2\coth(\hbar\omega/2k_BT)}\left(1 + \frac{\alpha}{\hbar}\langle\Psi_T(q)|\{p,q\}|\Psi_T(q)\rangle\right) \tag{53}$$

in the q representation. Because we average over the quasiparticle vacuum in formula (53), the first term in it vanishes. At the same time, it was shown by us {Su06} that

$$\langle\Psi_T(q)|\{\hat{p},\hat{q}\}|\Psi_T(q)\rangle = \hbar\alpha. \tag{54}$$

As a result, we obtain the expression for the effective internal energy of the quantum oscillator in the "thermal" QHB in the $\hbar k$D framework:

$$\mathcal{E}_{ef} = \frac{\hbar\omega}{2\coth(\hbar\omega/2k_BT)}(1 + \alpha^2) = \frac{\hbar\omega}{2}\coth\frac{\hbar\omega}{2k_BT} = \mathcal{E}_{Pl}, \tag{55}$$

that coincides with the formula (44). This means that the average energy of the quantum oscillator at $T \neq 0$ has the meaning of effective internal energy as a macroparameter in the case of equilibrium with the "thermal" QHB. As $T \to 0$, it passes to a similar quantity corresponding to equilibrium with the "cold" QHB.

Although final result (55) was totally expected, several significant conclusions follow from it.

1. In the $\hbar k$D, in contrast to calculating the internal energy in QSM, where all is defined by the probability density $\rho_T(q)$, the squared parameter α determining the phase of the wave function contributes significantly to the same expression, which indicates that the quantum ideology is used more consistently.

2. In the $\hbar k$D, the expression for $\coth(\hbar\omega/2k_BT)$ in formula (55) appears as an holistic quantity, while the contribution $\varepsilon_0 = \hbar\omega/2$ to the same formula (6) in QSM usually arises separately as an additional quantity without a thermodynamic meaning and is therefore often neglected.

3. In the $\hbar k$D, the operators $\hat{\mathcal{H}}$ and $\hat{N}_b$ do not commute. It demonstrates that the number of quasiparticles is not preserved, which is typical of the case of spontaneous

symmetry breaking. In our opinion, the proposed model of the QHB is a universal model of the environment with a stochastic influence on an object. Therefore, the manifestations of spontaneous symmetry breaking in nature must not be limited to superfluidity and superconductivity phenomena.

3.3 Schrödingerian as a stochastic action operator

The effective action as a macroparameter was postulated in the Section 1 in the framework of TEM by generalizing concepts of adiabatic invariants. In the $\hbar kD$ framework, we base our consistent microdescription of an object in thermal equilibrium on the model of the QHB described by a wave function of form (45).

Because the original statement of the $\hbar kD$ is the idea of the holistic stochastic influence of the QHB on the object, we introduce a new operator in the Hilbert space of microobject states to implement it. As leading considerations, we use an analysis of the right-hand side of the Schrödinger coordinate–momentum uncertainties relation in the saturated form {Su06}:

$$(\Delta p)^2 (\Delta q)^2 = |\widetilde{R}_{pq}|^2. \tag{56}$$

For not only a quantum oscillator in a heat bath but also any object, the complex quantity in the right-hand side of (56)

$$\widetilde{R}_{pq} = \langle \Delta p | \Delta q \rangle = \langle \, | \Delta \hat{p} \, \Delta \hat{q} | \, \rangle \tag{57}$$

has a double meaning. On one hand, it is the amplitude of the transition from the state $|\Delta q\rangle$ to the state $|\Delta p\rangle$; on the other hand, it can be treated as the mean of the Schrödinger quantum correlator calculated over an arbitrary state $|\,\rangle$ of some operator.

As is well known, the nonzero value of quantity (57) is the fundamental attribute of nonclassical theory in which the environmental stochastic influence on an object plays a significant role. Therefore, it is quite natural to assume that the averaged operator in the formula has a fundamental meaning. In view of dimensional considerations, we call it the stochastic action operator, or Schrödingerian

$$\hat{j} \equiv \Delta \hat{p} \Delta \hat{q}. \tag{58}$$

Of course, it should be remembered that the operators $\Delta \hat{q}$ and $\Delta \hat{p}$ do not commute and their product is a non-Hermitian operator.

To analyze further, following Schrödinger (1930) {DoMa87}, we can express the given operator in the form

$$\hat{j} = \frac{1}{2}\{\Delta \hat{p}, \Delta \hat{q}\} + \frac{1}{2}[\hat{p},\hat{q}] = \hat{\sigma} - i\hat{j}_0, \tag{59}$$

which allows separating the Hermitian part (the operator $\hat{\sigma}$) in it from the anti-Hermitian one, in which the Hermitian operator is

$$\hat{j}_0 = \frac{i}{2}[\hat{p},\hat{q}] \equiv \frac{\hbar}{2}\hat{I}. \tag{60}$$

It is easy to see that the mean $\sigma = \langle |\hat{\sigma}| \rangle$ of the operator $\hat{\sigma}$ resembles the expression for the standard correlator of coordinate and momentum fluctuations in classical probability theory; it transforms into this expression if the operators $\Delta \hat{q}$ and $\Delta \hat{p}$ are replaced with c-numbers. It reflects the contribution to the transition amplitude $\widetilde{R}_{pq}$ of the environmental stochastic influence. Therefore, we call the operator $\hat{\sigma}$ the external stochastic action operator in what follows. Previously, the possibility of using a similar operator was discussed by Bogoliubov

and Krylov (1939) as a quantum analogue of the classical action variable in the set of action–angle variables.

At the same time, the operators $\hat{j}_0$ and $\hat{j}$ were not previously introduced. The operator of form (60) reflects a specific peculiarity of the objects to be "sensitive" to the minimal stochastic influence of the "cold" vacuum and to respond to it adequately regardless of their states. Therefore, it should be treated as a minimal stochastic action operator. Its mean $\mathcal{J}_0 = \langle |\hat{j}_0| \rangle = \hbar/2$ is independent of the choice of the state over which the averaging is performed, and it hence has the meaning of the invariant eigenvalue of the operator $\hat{j}_0$.

This implies that in the given case, we deal with the universal quantity $\mathcal{J}_0$, which we call the minimal action. Its fundamental character is already defined by its relation to the Planck world constant $\hbar$. But the problem is not settled yet. Indeed, according to the tradition dating back to Planck, the quantity $\hbar$ is assumed to be called the elementary quantum of the action. At the same time, the factor $1/2$ in the quantity $\mathcal{J}_0$ plays a significant role, while half the quantum of the action is not observed in nature. Therefore, the quantities $\hbar$ and $\hbar/2$, whose dimensions coincide, have different physical meanings and must hence be named differently, in our opinion. From this standpoint, it would be more natural to call the quantity $\hbar$ the external quantum of the action.

Hence, the quantity $\hbar$ is the minimal portion of the action transferred to the object from the environment or from another object. Therefore, photons and other quanta of fields being carriers of fundamental interactions are first the carriers of the minimal action equal to $\hbar$. The same is also certainly related to phonons.

Finally, we note that only the quantity $\hbar$ is related to the discreteness of the spectrum of the quantum oscillator energy in the absence of the heat bath. At the same time, the quantity $\hbar/2$ has an independent physical meaning. It reflects the minimal value of stochastic influence of environment at $T = 0$, specifying by formula (42) the minimal value of the effective internal energy $\mathcal{E}_{ef}^0$ of the quantum oscillator.

3.4 Effective action in $(\hbar, k)$-dynamics

Now we can turn to the macrodescription of objects using their microdescription in the $\hbar k$D framework. It is easy to see that the mean $\tilde{\mathcal{J}}$ of the operator $\hat{j}$ of form (59) coincides with the complex transition amplitude $\tilde{R}_{pq}$ and, in thermal equilibrium, can be expressed as

$$\tilde{\mathcal{J}} = \langle \Psi_T(q) | \hat{j} | \Psi_T(q) \rangle = \sigma - i \mathcal{J}_0 = (\tilde{R}_{pq})_{ef}. \tag{61}$$

In what follows, we regard the modulus of the complex quantity $\tilde{\mathcal{J}}$,

$$|\tilde{\mathcal{J}}| = \sqrt{\sigma^2 + \mathcal{J}_0^2} = \sqrt{\sigma^2 + \frac{\hbar^2}{4}} \equiv \mathcal{J}_{ef} \tag{62}$$

as a new macroparameter and call it the effective action. It has the form

$$\mathcal{J}_{ef} = \frac{\hbar}{2} \coth \frac{\hbar \omega}{2 k_B T}, \tag{63}$$

that coincides with a similar quantity $\mathcal{J}_{ef}$ postulated as a fundamental macroparameter in TEM framework (see the Sect.1.) from intuitive considerations.

We now establish the interrelation between the effective action and traditional macroparameters. Comparing expression (63) for $|\tilde{\mathcal{J}}|$ with (55) for the effective internal

energy $\mathcal{E}_{ef}$, we can easily see that

$$\mathcal{E}_{ef} = \omega|\tilde{\mathcal{J}}| = \omega\mathcal{J}_{ef}. \tag{64}$$

In the high-temperature limit, where

$$\sigma \to \mathcal{J}_T = \frac{k_B T}{\omega} \gg \frac{\hbar}{2}, \tag{65}$$

relation (64) becomes

$$\mathcal{E} = \omega\mathcal{J}_T. \tag{66}$$

Boltzmann {Bol22} previously obtained this formula for macroparameters in CSM-based thermodynamics by generalizing the concept of adiabatic invariants used in classical mechanics.

Relation (64) also allows expressing the interrelation between the effective action and the effective temperature T_{ef} (8) in explicit form:

$$T_{ef} = \frac{\omega}{k_B}\mathcal{J}_{ef}. \tag{67}$$

This implies that

$$T_{ef}^0 = \frac{\omega}{k_B}\mathcal{J}_{ef}^0 = \frac{\hbar\omega}{2k_B} \neq 0, \tag{68}$$

where $\mathcal{J}_{ef}^0 \equiv \mathcal{J}_0$. Finally, we note that using formulas (56), (61)–(64), (46), and (47), we can rewrite the saturated Schrödinger uncertainties relation for the quantum oscillator for $T \neq 0$ as

$$\Delta p_{ef} \cdot \Delta q_{ef} = \mathcal{J}_{ef} = \frac{\mathcal{E}_{ef}}{\omega} = \frac{\hbar}{2}\coth\frac{\hbar\omega}{2k_B T}. \tag{69}$$

3.5 Effective entropy in the $(\hbar,k)$-dynamics

The possibility of introducing entropy in the $\hbar k$D is also based on using the wave function $\Psi_T(q)$ instead of the density operator. To define the entropy as the initial quantity, we take the formal expression

$$-k_B\left\{\int \rho(q)\log\rho(q)\,dq + \int \rho(p)\log\rho(p)\,dp\right\} \tag{70}$$

described in {DoMa87}. Here, $\rho(q) = |\Psi(q)|^2$ and $\rho(p) = |\Psi(p)|^2$ are the dimensional densities of probabilities in the respective coordinate and momentum representations.

Using expression (45) for the wave function of the quantum oscillator, we reduce $\rho(q)$ to the dimensionless form:

$$\tilde{\rho}(\tilde{q}) = \left[\frac{2\pi}{\delta}\coth\frac{\hbar\omega}{2k_B T}\right]^{-1}e^{-\tilde{q}^2/2}, \qquad \tilde{q}^2 = \frac{q^2}{(\Delta q_{ef})^2}, \tag{71}$$

where δ is an arbitrary constant. A similar expression for its Fourier transform $\tilde{\rho}(\tilde{p})$ differs by only replacing q with p.

Using the dimensionless expressions, we propose to define entropy in the $\hbar k$D framework by the equality

$$S_{qp} = -k_B\left\{\int \tilde{\rho}(\tilde{q})\log\tilde{\rho}(\tilde{q})\,d\tilde{q} + \int \tilde{\rho}(\tilde{p})\log\tilde{\rho}(\tilde{p})\,d\tilde{p}\right\}. \tag{72}$$

Substituting the corresponding expressions for $\tilde{\rho}(\tilde{q})$ and $\tilde{\rho}(\tilde{p})$ in (72), we obtain

$$S_{qp} = k_B \left\{ \left(1 + \log \frac{2\pi}{\delta} \right) + \log \coth \frac{\hbar \omega}{2k_B T} \right\}. \tag{73}$$

Obviously, the final result depends on the choice of the constant δ.

Choosing $\delta = 2\pi$, we can interpret expression (73) as the quantum-thermal entropy or, briefly, the QT entropy S_{QT} because it coincides exactly with the effective entropy S_{ef} (15). This ensures the consistency between the main results of our proposed micro- and macrodescriptions, i.e. $\hbar k$D and TEM, and their correspondence to experiments.

We can approach the modification of original formal expression (70) in another way. Combining both terms in it, we can represent it in the form

$$- k_B \int d\varepsilon \, W(\varepsilon) \log W(\varepsilon). \tag{74}$$

It is easy to see that $W(\varepsilon)$ is the Wigner function for the quantum oscillator in the QHB:

$$W(\varepsilon) = \{2\pi \Delta q \Delta p\}^{-1} \exp \left\{ - \frac{p^2}{2(\Delta p)^2} - \frac{q^2}{2(\Delta q)^2} \right\} = \frac{\omega}{2\pi k_B T_{ef}} e^{-\varepsilon/k_B T_{ef}}. \tag{75}$$

After some simple transformations the expression (74) takes also the form $S_{ef} = S_{QT}$.

Modifying expressions (70) (for $\delta = 2\pi$) or (74) in the $\hbar k$D framework thus leads to the expression for the QT, or effective, entropy of form (15). From the microscopic standpoint, they justify the expression for the effective entropy as a macroparameter in MST. We note that the traditional expression for entropy in QSM-based thermodynamics turns out to be only a quasiclassical approximation of the QT, or effective entropy.

3.6 Some thermodynamics relations in terms of the effective action

The above presentation shows that using the $\hbar k$D developed here, we can introduce the effective action $\mathcal{J}_{ef}$ as a new fundamental macroparameter. The advantage of this macroparameter is that in the given case, it has a microscopic preimage, namely, the stochastic action operator $\hat{j}$, or Schrödingerian. Moreover, we can in principle express the main macroparameters of objects in thermal equilibrium in terms of it. As is well known, temperature and entropy are the most fundamental of them. It is commonly accepted that they have no microscopic preimages but take the environment stochastic influence on the object generally into account. In the traditional presentation, the temperature is treated as a "degree of heating," and entropy is treated as a "measure of system chaos."

If the notion of effective action is used, these heuristic considerations about T_{ef} and S_{ef} can acquire an obvious meaning. For this, we turn to expression (67) for T_{ef}, whence it follows that the effective action is also an *intensive* macroparameter characterizing the stochastic influence of the QHB. In view of this, the zero law of MST can be rewritten as

$$\mathcal{J}_{ef} = (\mathcal{J}_{ef})_0 \pm \delta \mathcal{J}_{ef}, \tag{76}$$

where $(\mathcal{J}_{ef})_0$ is the effective action of a QHB and $\mathcal{J}_{ef}$ and $\delta \mathcal{J}_{ef}$ are the means of the effective reaction of an object and its fluctuation. The state of thermal equilibrium can actually be described in the sense of Newton, assuming that "the stochastic action is equal to the stochastic counteraction" in such cases.

We now turn to the effective entropy S_{ef}. In the absence of a mechanical contact, its differential in MST is

$$dS_{ef} = \frac{\delta Q_{ef}}{T_{ef}} = \frac{d\mathcal{E}_{ef}}{T_{ef}}. \tag{77}$$

Substituting the expressions for effective internal energy (64) and effective temperature (67) in this relation, we obtain

$$dS_{ef} = k_B \frac{\omega\, d\mathcal{J}_{ef}}{\omega \mathcal{J}_{ef}} = k_B \cdot d\left(\log \frac{\mathcal{J}_{ef}}{\mathcal{J}_{ef}^0}\right) = dS_{QT}. \tag{78}$$

It follows from this relation that the effective or QT entropy, being an *extensive* macroparameter, can be also expressed in terms of $\mathcal{J}_{ef}$.

As a result, it turns out that two qualitatively different characteristics of thermal phenomena on the macrolevel, namely, the effective temperature and effective entropy, embody the presence of two sides of stochastization the characteristics of an object in nature in view of the contact with the QHB. At any temperature, they can be expressed in terms of the same macroparameter, namely, the effective action $\mathcal{J}_{ef}$. This macroparameter has the stochastic action operator, or Schrödingerian simultaneously dependent on the Planck and Boltzmann constants as a microscopic preimage in the $\hbar k$D.

4. Theory of effective macroparameters fluctuations and their correlation

In the preceding sections we considered effective macroparameters as random quantities but the subject of interest were only problems in which the fluctuations of the effective temperature and other effective object macroparameters can be not taken into account. In given section we consistently formulate a noncontradictory theory of quantum-thermal fluctuations of effective macroparameters (TEMF) and their correlation. We use the apparatus of two approaches developed in sections 2 and 3 for this purpose.

This theory is based on the rejection of the classical thermostat model in favor of the quantum one with the distribution modulus $\Theta_{qu} = k_B T_{ef}$. This allows simultaneously taking into account the quantum and thermal stochastic influences of environment describing by effective action. In addition, it is assumed that some of macroparameters fluctuations are obeyed the nontrivial uncertainties relations. It appears that correlators of corresponding fluctuations are proportional to effective action $\mathcal{J}_{ef}$.

4.1 Inapplicability QSM-based thermodynamics for calculation of the macroparameters fluctuations

As well known, the main condition of applicability of thermodynamic description is the following inequality for relative dispersion of macroparameter A_i :

$$\frac{(\Delta A_i)^2}{\langle A_i\rangle^2} \leqslant 1, \tag{79}$$

where

$$(\Delta A_i)^2 \equiv \langle(\delta A_i)^2\rangle = \langle A_i^2\rangle - \langle A_i\rangle^2$$

is the dispersion of the quantity A_i.

In the non-quantum version of statistical thermodynamics, the expressions for macroparameters dispersions can be obtained. So, for dispersions of the temperature

and the internal energy of the object for $V = const$ we have according to Einstein {LaLi68}

$$(\Delta T)^2 = \frac{1}{k_B C_V} \Theta_{cl}^2 = \frac{k_B}{C_V} T^2 \text{ and } (\Delta \mathcal{E})^2 = \frac{C_V}{k_B} \Theta_{cl}^2 = k_B C_V T^2, \tag{80}$$

where $C_V = \frac{\partial \langle \mathcal{E} \rangle}{\partial T}\big|_V$ is the heat capacity for the constant volume. At high temperatures the condition (79) is satisfies for any macroparameters and any objects including the classical oscillator.

For its internal energy $\mathcal{E} = \langle \varepsilon \rangle = k_B T$ with account $C_V = k_B$ we obtain its dispersion

$$(\Delta \mathcal{E})^2 = k_B C_V T^2 = k_B^2 T^2 = \mathcal{E}^2. \tag{81}$$

So, the condition (79) is valid for $\mathcal{E}$ and this object can also be described in the framework of thermodynamics.

For the account of quantum effects in QSM-based thermodynamics instead of (80) are used the following formulae

$$(\Delta T)^2 = 0 \text{ and } (\Delta \mathcal{E}_{qu})^2 = k_B (C_V)_{qu} T^2. \tag{82}$$

The difference is that instead of C_V, it contains

$$(C_V)_{qu} = \frac{\partial \mathcal{E}_{qu}}{\partial T}\bigg|_V,$$

where $\mathcal{E}_{qu} = \langle \varepsilon_{qu} \rangle$ is the internal energy of the object calculated in the QSM framework. For a quantum oscillator in this case we have

$$\langle \mathcal{E}_{qu} \rangle = \frac{\hbar \omega}{\exp\{2\varkappa \frac{\omega}{T}\} - 1} = \frac{\hbar \omega}{2} \cdot \frac{\exp\{-\varkappa \frac{\omega}{T}\}}{\sinh(\varkappa \frac{\omega}{T})}, \tag{83}$$

and its heat capacity is

$$(C_V)_{qu} = k_B \left(\frac{\hbar \omega}{k_B T}\right)^2 \frac{\exp\{2\varkappa \frac{\omega}{T}\}}{(\exp\{2\varkappa \frac{\omega}{T}\} - 1)^2} = k_B \left(\varkappa \frac{\omega}{T}\right)^2 \frac{1}{\sinh^2(\varkappa \frac{\omega}{T})}. \tag{84}$$

According to general formula (82), the dispersion of the quantum oscillator internal energy has the form

$$(\Delta \mathcal{E}_{qu})^2 = k_B (C_V)_{qu} T^2 = \left(\frac{\hbar \omega}{2}\right)^2 \cdot \frac{1}{\sinh^2(\varkappa \frac{\omega}{T})} =$$
$$= \hbar \omega \langle \mathcal{E}_{qu} \rangle + \langle \mathcal{E}_{qu} \rangle^2 = \exp\{2\varkappa \frac{\omega}{T}\} \langle \mathcal{E}_{qu} \rangle^2, \tag{85}$$

and the relative dispersion of its energy is

$$\frac{(\Delta \mathcal{E}_{qu})^2}{\langle \mathcal{E}_{qu} \rangle^2} = \frac{\hbar \omega}{\langle \mathcal{E}_{qu} \rangle} + 1 = \exp\{2\varkappa \frac{\omega}{T}\}. \tag{86}$$

We note that in expression (83) the zero-point energy $\varepsilon_0 = \hbar \omega / 2$ is absent. It means that the relative dispersion of internal energy stimulating by thermal stochastic influence are only the subject of interest. So, we can interpret this calculation as a quasiclassical approximation.

A similar result exists for the relative dispersion of the energy of thermal radiation in the spectral interval $(\omega, \omega + \Delta\omega)$ for the volume V :

$$\frac{(\Delta\mathcal{E}_\omega)^2}{\langle\mathcal{E}_\omega\rangle^2} = \frac{\hbar\omega}{\langle\mathcal{E}_\omega\rangle} + \frac{\pi^2 c^3}{V\omega^2\Delta\omega} = \frac{\pi^2 c^3}{V\omega^2\Delta\omega}\exp\{2\varkappa\frac{\omega}{T}\}. \tag{87}$$

We can see that at $T \to 0$ expressions (86) and (87) tend to infinity. However, few people paid attention to the fact that thereby the condition (79) of the applicability of the thermodynamic description does not satisfy. A.I. Anselm {An73} was the only one who has noticed that ordinary thermodynamics is inapplicable as the temperature descreases. We suppose that in this case instead of QSM-based thermodynamics can be fruitful MST based on $\hbar k$D.

4.2 Fluctuations of the effective internal energy and effective temperature

To calculate dispersions of macroparameters in the quantum domain, we use MST instead of QSM-based thermodynamics in 4.2 and 4.3, i.e., we use the macrotheory described in Sect.1. It is based on the Gibbs distribution in the effective macroparameters space {Gi60}

$$dW(\mathcal{E}) = \rho(\mathcal{E})d\mathcal{E} = \frac{1}{k_B T_{ef}}\exp\{-\frac{\mathcal{E}}{k_B T_{ef}}\}d\mathcal{E}. \tag{88}$$

Here, T_{ef} is the effective temperature of form (8), simultaneously taking the quantum–thermal effect of the QHB into account and $\mathcal{E}$ is the random object energy to which the conditional frequency ω can be assigned at least approximately.

Using distribution (88), we find the expression for the effective internal energy of the object coinciding with the Planck formula

$$\mathcal{E}_{ef} = \langle\varepsilon_{qu}\rangle = \int\mathcal{E}\rho(\mathcal{E})d\mathcal{E} = k_B T_{ef} \equiv \mathcal{E}_{Pl} = \frac{\hbar\omega}{2}\coth\varkappa\frac{\omega}{T}, \tag{89}$$

the average squared effective internal energy

$$\langle\mathcal{E}_{ef}^2\rangle = \int\mathcal{E}^2\rho(\mathcal{E})d\mathcal{E} = 2\langle\mathcal{E}_{ef}\rangle^2, \tag{90}$$

and the dispersion of the effective internal energy

$$(\Delta\mathcal{E}_{ef})^2 = \langle\mathcal{E}_{ef}^2\rangle - \langle\mathcal{E}_{ef}\rangle^2 = \langle\mathcal{E}_{ef}\rangle^2. \tag{91}$$

It is easy to see that its relative dispersion is unity, so that condition (79) holds in this case.

For the convenience of the comparison of the obtained formulae with the non quantum version of ST {LaLi68}, we generalize the concept of heat capacity, introducing the effective heat capacity of the object

$$(C_V)_{ef} \equiv \frac{\partial\langle\mathcal{E}_{ef}\rangle}{\partial T_{ef}} = k_B. \tag{92}$$

This allows writing formula (91) for the dispersion of the internal energy in a form that is similar to formula (83), but the macroparameters are replaced with their effective analogs in this case:

$$(\Delta\mathcal{E}_{ef})^2 = k_B(C_V)_{ef}T_{ef}^2 = k_B^2 T_{ef}^2. \tag{93}$$

It should be emphasized that we assumed in all above-mentioned formulae in Sect.4 that $T_{ef} = (T_{ef})_0$ and $T = T_0$, where $(T_{ef})_0$ and T_0 are the effective and Kelvin temperature of the QHB correspondingly.

Indeed, in the macrotheory under consideration, we start from the fact that the effective object temperature T_{ef} also experiences fluctuations. Therefore, the zero law according to (67) and (76) becomes

$$T_{ef} = T_{ef}^0 \pm \delta T_{ef}, \tag{94}$$

where δT_{ef} is the fluctuation of the effective object temperature. According to the main MST postulate, the form of the expression for the dispersion of the effective object temperature is similar to that of expression (80):

$$(\Delta T_{ef})^2 = \frac{k_B}{(C_V)_{ef}} T_{ef}^2 = T_{ef}^2, \tag{95}$$

so that the relative dispersion of the effective temperature also obeys condition (79).

To compare the obtained formulae with those in QSM-based thermodynamics, we represent dispersion of the effective internal energy (93) in the form

$$(\Delta \mathcal{E}_{ef})^2 = \left(\frac{\hbar\omega}{2}\right)^2 (\coth \varkappa \frac{\omega}{T})^2 = \left(\frac{\hbar\omega}{2}\right)^2 \cdot [1 + \sinh^{-2}(\varkappa \frac{\omega}{T})]. \tag{96}$$

The comparison of formula (96) with expression (85), where the heat capacity has form (84), allows writing the second term in (96) in the form resembling initial form (81)

$$(\Delta \mathcal{E}_{ef})^2 = (\frac{\hbar\omega}{2})^2 + k_B (C_V)_{qu} T^2. \tag{97}$$

However, in contrast to formula (85), the sum in it is divided into two terms differently. Indeed, the first term in formula (97) can be written in the form

$$(\frac{\hbar\omega}{2})^2 = \frac{\hbar}{2} \rho_\omega(\omega,0)\omega^2, \tag{98}$$

where

$$\rho_\omega(\omega,0) \equiv \left. \frac{\partial \langle \mathcal{E}_{ef} \rangle}{\partial \omega} \right|_{T=0} = \frac{\hbar}{2}$$

is the spectral density of the effective internal energy at $T = 0$. Then formula (97) for the dispersion of the effective internal energy acquires the form generalizing formula (85):

$$(\Delta \mathcal{E}_{ef})^2 = \frac{\hbar}{2} \rho_\omega(\omega,0)\omega^2 + k_B [C_V(\omega,T)]_{qu} T^2. \tag{99}$$

It is of interest to note that in contrast to formula (85) for the quantum oscillator or a similar formula for thermal radiation, an additional term appears in formula (99) and is also manifested in the cold vacuum. The symmetric form of this formula demonstrates that the concepts of characteristics, such as frequency and temperature, are similar, which is manifested in the expression for the minimal effective temperature $T_{ef}^0 = \varkappa\omega$. The corresponding analogies between the world constant $\hbar/2$ and k_B and also between the characteristic energy "densities" ρ_ω and $(C_V)_{qu}$ also exist.

In the limit $T \to 0$, only the first term remains in formula (99), and, as a result,

$$(\Delta \mathcal{E}_{ef}^0)^2 = (\mathcal{E}_{ef}^0)^2 = (\frac{\hbar\omega}{2})^2 \neq 0. \tag{100}$$

In our opinion, we have a very important result. This means that zero-point energy is "smeared", i.e. it has a non-zero width. It is natural that the question arises as to what is the reason for the fluctuations of the effective internal energy in the state with $T = 0$. This is because the peculiar stochastic thermal influence exists even at zero Kelvin temperature due to $T_{ef} \neq 0$. In this case the influence of "cold" vacuum in the form (100) is equivalent to $k_B T_{ef}^0 / \omega$. In contrast to this, $(\Delta \mathcal{E}_{qu})^2 \to 0$, as $T \to 0$ in QSM-based thermodynamics, because the presence of the zero point energies is taken into account not at all in this theory.

4.3 Correlation between fluctuations and the uncertainties relations for effective macroparameters

Not only the fluctuations of macroparameters, but also the correlation between them under thermal equilibrium play an important role in MST. This correlation is reflected in correlators contained in the uncertainties relations (UR) of macroparameters {Su05}

$$\Delta A_i \Delta A_j \geqslant \langle \delta A_i, \delta A_j \rangle, \tag{101}$$

where the uncertainties ΔA_i and ΔA_j on the left and the correlator on the right must be calculated independently. If the right side of (101) is not equal to zero restriction on the uncertainties arise.

We now pass to analyzing the correlation between the fluctuations of the effective macroparameters in thermal equilibrium. We recall that according to main MST postulate, the formulae for dispersions and correlators remain unchanged, but all macroparameters contained in them are replaced with the effective ones: $A_i \to (A_{ef})_i$.

a). Independent effective macroparameters

Let us consider a macrosystem in the thermal equilibrium characterizing in the space of effective macroparameters by the pair of variables T_{ef} and V_{ef}. Then the probability density of fluctuations of the effective macroparameters becomes {LaLi68}, {An73}

$$W(\delta T_{ef}, \delta V_{ef}) = C \exp \left\{ -\frac{1}{2} \left(\frac{\delta T_{ef}}{\Delta T_{ef}} \right)^2 - \frac{1}{2} \left(\frac{\delta V_{ef}}{\Delta V_{ef}} \right)^2 \right\}. \tag{102}$$

Here, C is the normalization constant, the dispersion of the effective temperature $(\Delta T_{ef})^2$ has form (95), and the dispersion of the effective volume δV_{ef} is

$$(\Delta V_{ef})^2 = -k_B T_{ef} \left. \frac{\partial V_{ef}}{\partial P_{ef}} \right|_{T_{ef}}. \tag{103}$$

We note that both these dispersions are nonzero for any T.

Accordingly to formula (102) the correlator of these macroparameters $\langle \delta T_{ef}, \delta V_{ef} \rangle = 0$. This equality confirms the independence of the fluctuations of the effective temperature and volume. Hence it follows that the UR for these quantities has the form $\Delta T_{ef} \Delta V_{ef} \geqslant 0$, i.e., no additional restrictions on the uncertainties ΔT_{ef} and ΔV_{ef} arise from this relation.

b). Conjugate effective macroparameters

As is well known, the concept of conjugate quantities is one of the key concepts in quantum mechanics. Nevertheless, it is also used in thermodynamics but usually on the basis of

heuristic considerations. Without analyzing the physical meaning of this concept in MST (which will be done in 4.4), we consider the specific features of correlators and URs for similar pairs of effective macroparameters.

Based on the first law of thermodynamics, Sommerfeld emphasized {So52}] that entropy is a macroparameter conjugate to temperature. To obtain the corresponding correlator, we calculate the fluctuation of the effective entropy S_{ef}:

$$\delta S_{ef} = \left.\frac{\partial S_{ef}}{\partial T_{ef}}\right|_{V_{ef}} \delta T_{ef} + \left.\frac{\partial S_{ef}}{\partial V_{ef}}\right|_{T_{ef}} \delta V_{ef} = \frac{(C_V)_{ef}}{T_{ef}} \delta T_{ef} + \left.\frac{\partial P_{ef}}{\partial T_{ef}}\right|_{V_{ef}} \delta V_{ef} \qquad (104)$$

In the calculation of the correlator of fluctuations of the macroparameters δS_{ef} and δT_{ef} using distribution (102), the cross terms vanish because of the independence of the quantities δV_{ef} and δT_{ef}. As a result, the correlator contains only one term proportional to $(\Delta T_{ef})^2$ so that $\langle \delta S_{ef}, \delta T_{ef} \rangle$ becomes

$$\langle \delta S_{ef}, \delta T_{ef} \rangle = \frac{(C_V)_{ef}}{T_{ef}} (\Delta T_{ef})^2 = k_B T_{ef}. \qquad (105)$$

We note that, the obtained expression depends linearly on T_{ef}.

To analyze the desired UR, we find the dispersion $(\Delta S_{ef})^2$, using distribution (102):

$$(\Delta S_{ef})^2 = \left(\frac{(C_V)_{ef}}{T_{ef}}\right)^2 (\Delta T_{ef})^2 + \left(\left.\frac{\partial P_{ef}}{\partial T_{ef}}\right|_{V_{ef}}\right)^2 (\Delta V_{ef})^2, \qquad (106)$$

where ΔT_{ef} and ΔV_{ef} are defined by formulas (95) and (103). This expression can be simplified for $V_{ef} = const$. Thus, if (92) and (95) are taken into account, the uncertainty ΔS_{ef} becomes

$$\Delta S_{ef} = \frac{(C_V)_{ef}}{T_{ef}} (\Delta T_{ef}) = k_B. \qquad (107)$$

As a result, the uncertainties product in the left-hand side of the UR has the form

$$(\Delta S_{ef})(\Delta T_{ef}) = k_B T_{ef}. \qquad (108)$$

Combining formulas (108) and (105), we finally obtain the "effective entropy–effective temperature" UR in the form of an equality

$$(\Delta S_{ef})(\Delta T_{ef}) = k_B T_{ef} = \langle \delta S_{ef}, \delta T_{ef} \rangle. \qquad (109)$$

In the general case, for $V_{ef} \neq const$, the discussed UR implies the inequality

$$\Delta S_{ef} \Delta T_{ef} \geqslant k_B T_{ef}. \qquad (110)$$

In other words, the uncertainties product in this case is restricted to the characteristic of the QHB, namely, its effective temperature, which does not vanish in principle. This is equivalent to the statement that the mutual restrictions imposed on the uncertainties ΔS_{ef} and ΔT_{ef} are governed by the state of thermal equilibrium with the environment. Analogical result is valid for conjugate effective macroparameters the pressure P_{ef} and V_{ef}.

4.4 Interrelation between the correlators of conjugate effective macroparameters fluctuations and the stochastic action. The second holistic stochastic-action constant

To clarify the physical meaning of the correlation of macroparameters fluctuations we turn to results of the sections 2 and 3. In this case, we proceed from the Bogoliubov idea, according to which only the environmental stochastic influence can be the reason for the appearance of a nontrivial correlation between fluctuations of both micro and macroparameters.

We recall that the effective action $\mathcal{J}_{ef}$ in MST which is connected with the Schrödingerian in $\hbar kD$ is a characteristic of stochastic influence. Its definition in formula (62) was related to the quantum correlator of the canonically conjugate quantities, namely, the coordinate and momentum in the thermal equilibrium state. In this state, the corresponding UR is saturated {Su06}:

$$\Delta p_{ef}\Delta q_{ef} \equiv \mathcal{J}_{ef}, \tag{111}$$

where uncertainties are

$$\Delta p_{ef} = \sqrt{m\omega}\sqrt{\mathcal{J}_{ef}} \ \ \text{and} \ \ \Delta q_{ef} = \frac{1}{\sqrt{m\omega}}\sqrt{\mathcal{J}_{ef}}.$$

We stress that in this context, the quantities p_{ef} and q_{ef} also have the meaning of the effective macroparameters, which play an important role in the theory of Brownian motion.

We show that correlator of the effective macroparameters (105) introduced above also depend on $\mathcal{J}_{ef}$. We begin our consideration with the correlator of "effective entropy–effective temperature" fluctuations. Using (110), we can write relation (105) in the form

$$\langle \delta S_{ef}, \delta T_{ef}\rangle = \omega\mathcal{J}_{ef} \ \ \text{or} \ \ \langle \delta S_{ef}, \delta \mathcal{J}_{ef}\rangle = k_B\mathcal{J}_{ef}. \tag{112}$$

Thus, we obtain two correlators of different quantities. They depend linearly on the effective action $\mathcal{J}_{ef}$; so, they are equivalent formally.

However, the pair of correlators in formula (112) is of interest from the physical point of view because their external identity is deceptive. In our opinion, the second correlator is more important because it reflects the interrelation between the environmental stochastic influence in the form $\delta\mathcal{J}_{ef}$ and the response of the object in the form of entropy fluctuation δS_{ef} to it.

To verify this, we consider the limiting value of this correlator as $T \to 0$ that is equal to the production

$$k_B\mathcal{J}_{ef}^0 = k_B\frac{\hbar}{2} \equiv \widetilde{\varkappa}, \tag{113}$$

where $\widetilde{\varkappa}$ is the *second* holistic stochastic action constant differing from the *first* one $\varkappa = \hbar/2k_B$. In the macrotheory, it is a minimal restriction on the uncertainties product of the effective entropy and the effective action:

$$\Delta S_{ef}^0\Delta\mathcal{J}_{ef}^0 = k_B\frac{\hbar}{2} = \widetilde{\varkappa} \neq 0. \tag{114}$$

The right-hand side of this expression contains the combination of the world constants k_B and $\frac{\hbar}{2}$, which was not published previously.

We compare expression (114) with the limiting value of the Schrödinger quantum correlator for the "coordinate–momentum" microparameters {Su06}, which are unconditionally assumed to be conjugate. In the microtheory, it is a minimum restriction on the product of the uncertainties Δp and Δq and is equal to

$$\Delta p_0\Delta q_0 = \mathcal{J}_{ef}^0 = \frac{\hbar}{2},$$

i.e., it also depends only on the world constant. Accordingly, convincing arguments used to admit that $\mathcal{J}_{ef}$ and S_{ef} are conjugate macroparameters appear.

Summarizing the above considerations, we formulate the criterion that allows us to independently estimate, what pair of the macroparameters can be considered conjugate.In our opinion, it reduces to the following conditions: a). the correlator of their fluctuations depends on $\mathcal{J}_{ef}$ linearly, and b). the minimum restriction on the uncertainties product is fixed by either one of the world constants $\frac{1}{2}\hbar$ and k_B or their product.

We note that the correlators of conjugate macroparameters fluctuations vanish in the case of the classical limit where environmental stochastic influence of quantum and thermal types are not taken into account. In this case, the corresponding quantities can be considered independent, the URs for them become trivial, and any restrictions on the values of their uncertainties vanish.

4.5 Transport coefficients and their interrelation with the effective action

We now turn to the analysis of transport coefficients. It follows from the simplest considerations of kinetic theory that all these coefficients are proportional to each other. We show below, what is the role of the effective action $\mathcal{J}_{ef}$ in this interrelation.

As we established {Su06}, "coordinate–momentum" UR (111) for the quantum oscillator in a thermostat can be written in the form

$$\Delta p_{ef}\Delta q_{ef} = mD_{ef}. \tag{115}$$

Then, for the effective self-diffusion coefficient with account (111), we have the expression

$$D_{ef} = \frac{\mathcal{J}_{ef}}{m}. \tag{116}$$

We now take into account the relation between the effective shear viscosity coefficient η_{ef} and the coefficient D_{ef}. We then obtain

$$\eta_{ef} = D_{ef}\rho_m = \frac{\mathcal{J}_{ef}}{V}, \tag{117}$$

where ρ_m is the mass density.

In our opinion, the ratio of the heat conductivity to the electroconductivity contained in the Wiedemann–Franz law is also of interest:

$$\frac{\lambda}{\sigma} = \gamma(\frac{k_B}{e})^2 T = \gamma\frac{k_B}{e^2}(k_B T), \tag{118}$$

where γ is a numerical coefficient. Obviously, the presence of the factor $k_B T$ in it implies that the classical heat bath model is used.

According to the main MST postulate, the generalization of this law to the QHB model must have the form

$$\frac{\lambda_{ef}}{\sigma_{ef}} = \gamma(\frac{k_B}{e})^2 T_{ef} = \gamma\frac{k_B}{e^2}(k_B T_{ef}) = \gamma\frac{k_B}{e^2}(k_B T_{ef}^0)\coth\varkappa\frac{\omega}{T}. \tag{119}$$

It is probable that this formula, which is also valid at low temperatures, has not been considered in the literature yet. As $T \to 0$, from (119), we obtain

$$\left(\frac{\lambda_{ef}^0}{\sigma_{ef}^0}\right) = \gamma(\frac{k_B}{e})^2 T_{ef}^0 = \gamma\frac{\omega}{e^2}(k_B\frac{\hbar}{2}) = \gamma\frac{\omega}{e^2}\widetilde{\varkappa}, \tag{120}$$

where $T_{ef}^0 = \varkappa\omega$, and the constant $\tilde{\varkappa}$ coincides with the correlator $\langle\delta S_{ef}^0, \delta\mathcal{J}_{ef}^0\rangle$ according to (114). We assume that the confirmation of this result by experiments is of interest.

5. Conclusion

So, we think that QSM and non-quantum version of ST as before keep their concernment as the leading theories in the region of their standard applications.

But as it was shown above, MST allows filling gaps in domains that are beyond of these frameworks. MST is able to be a ground theory at calculation of effective macroparameters and, their dispersions and correlators at low temperatures.

In the same time, MST can be also called for explanation of experimental phenomena connected with behavior of the ratio "shift viscosity to the volume density of entropy" in different mediums. This is an urgent question now for describing of nearly perfect fluids features.

In additional, the problem of zero-point energy smearing is not solved in quantum mechanics. In this respect MST can demonstrate its appreciable advantage because it from very beginning takes the stochastic influence of cold vacuum into account. This work was supported by the Russian Foundation for Basic Research (project No. 10-01-90408).

6. References

Anselm, A.I. (1973). *The Principles of Statistical Physics and Thermodynamics*, Nauka, ISBN 5-354-00079-3 Moscow

Bogoliubov, N.N. (1967). *Lectures on Quantum Statistics*, Gordon & Breach, Sci.Publ.,Inc. New York V.1 Quantum Statistics. 250 p.

Boltzmann, L. (1922). *Vorlesungen über die Prinzipien der Mechanik*, Bd.2, Barth, Leipzig

Dodonov, V.V. & Man'ko, V.I. (1987). Generalizations of Unsertainties Relations in Quantum Mechanics. *Trudy Lebedev Fiz. Inst.*, Vol.183, (September 1987) (5-70), ISSN 0203-5820

Gibbs, J.W. (1960). *Elementary Principles in Statistical Mechanics*, Dover, ISBN 1-881987-17-5 New York

Landau, L.D. & Lifshits E.M. (1968). *Course of Theoretical Physics*, V.5, Pergamon Press, ISBN 5-9221-00055-6 Oxford

Sommerfeld, A. (1952). *Thermodynamics and Statistical Physics*, Cambridge Univ. Press, ISBN 0-521-28796-0 Cambridge

Sukhanov A.D. (1999). On the Global Interrelation between Quantum Dynamics and Thermodynamics, *Proceedings of 11-th Int. Conf "Problems of Quantum Field Theory"*, pp. 232-236, ISBN 5-85165-523-2, Dubna, July 1998, JINR, Dubna

Sukhanov, A.D. (2005). Einstein's Statistical-Thermodynamic Ideas in a Modern Physical Picture of the World. *Phys. Part. Nucl.*, Vol.36, No.6, (December 2005) (667-723), ISSN 0367-2026

Sukhanov, A.D. (2006). Schrödinger Uncertainties Relation for Quantum Oscillator in a Heat Bath. *Theor. Math. Phys*, Vol.148, No.2, (August 2006) (1123-1136), ISSN 0564-6162

Sukhanov, A.D. (2008). Towards a Quantum Generalization of Equilibrium Statistical Thermodynamics: Effective Macroparameters *Theor. Math. Phys*, Vol.153, No.1, (January 2008) (153-164), ISSN 0564-6162

Sukhanov, A.D. & Golubjeva O.N. (2009). Towards a Quantum Generalization of Equilibrium Statistical Thermodynamics: $(\hbar - k)$– Dynamics *Theor. Math. Phys*, Vol.160, No.2, (August 2009) (1177-1189), ISSN 0564-6162
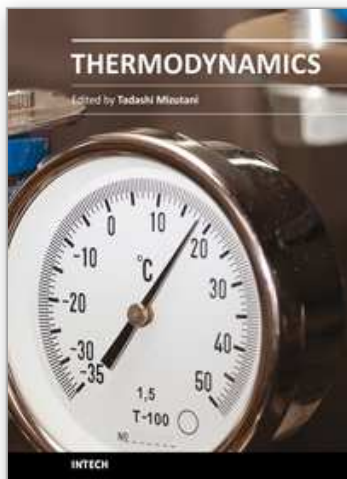
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