Finite Element Modelling of Sound Transmission Loss in Reflective Pipe

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1. Introduction

Acoustics is the physics of sound. Sound is the sensation, as detected by the ear, of very small rapid changes in the air pressure above and below a static value. This static value is atmospheric pressure (about 100,000 Pascals), which varies slowly. Associated with a sound pressure wave is a flow of energy. Physically, sound in air is a longitudinal wave where the wave motion is in the direction of the movement of energy. The wave crests are the pressure maxima, while the troughs represent the pressure minima (Comsol, 2007a).
Sound results when the air is disturbed by some source. An example is a vibrating object, such as a speaker cone in a hi-fi system. It is possible to see the movement of a bass speaker cone when it generates sound at a very low frequency. As the cone moves forward, it compresses the air in front of it, causing an increase in air pressure. Then it moves back past its resting position and causes a reduction in air pressure. This process continues, radiating a wave of alternating high and low pressure at the speed of sound.
Plane waves of constant frequency propagating through bulk materials have amplitudes that typically decrease exponentially with increasing propagation distance, such that the magnitude of the complex pressure amplitude varies as (Rossing, 2007)

$$|p(x)| = |p(0)|e^{-\alpha x} . \tag{1}$$

The quantity α is the plane wave attenuation coefficient and has units of nepers per meter (*Np/m*). It is an intrinsic frequency-dependent property of the material. This exponential decrease of amplitude is called attenuation or absorption of sound and is associated with the transfer of acoustic energy to the internal energy of the material. If $|p|^2$ decreases to a tenth of its original value, it is said to have decreased by 10 decibels (dB), so an attenuation constant of α nepers per meter is equivalent to an attenuation constant of [20/(ln 10)] α decibels per meter, or 8.6859α decibels per meter.
The attenuation of sound due to the classical processes of viscous energy absorption and thermal conduction is derivable from the dissipative wave equation given previously for the acoustics mode (Rossing, 2007). In air, the relaxation processes that affect sound attenuation

are those associated with the (quantized) internal vibrations of the diatomic molecules O_2 and N_2. The presence of the duct walls affects the attenuation of sound and usually causes the attenuation coefficient to be much higher than for plane waves in open space.
The problem of steady-state diffraction or transmission of sound energy by an aperture in a plane wall has attracted much attention in the literature for many years (Wilson and Soroka, 1965). Exact solutions are restricted to a few cases where the aperture geometry is simple and may be conveniently described in a coordinate system in which the wave equation becomes separable, or may allow the postulation of a set of velocity potentials that can be made to fit the boundaries. The theoretical methods available for approximating these solutions are valid for only a limited range of frequencies or wavelengths. The exact solutions are also of limited range in application because of practical difficulties in evaluating the infinite series occurring in the solutions. A frequency dependent model for the transmission loss of hard-walled pipe or duct is important for accurate estimates of more complicated acoustic systems (for instance, silencer-pipe connections).
Theoretical solutions of two- and three-dimensional acoustic wave propagation in hard-walled (Cummings, 1974; Rostafinski, 1974) and acoustically lined (Ko & Ho, 1977; Ko, 1979) ducts with rectangular cross-section have been presented by others researchers. In these studies, cylindrical coordinate system has been used to describe the curved duct. The theoretical solution for a circular section curved duct has not been achieved yet due to the mathematical difficulties encountered in the solution of the wave equation.
In paper Wilson and Soroka (1965) an approximate solution for the diffraction of a planes sound wave incident normally on a circular aperture in a plane rigid wall of finite thickness is obtained by postulating rigid, massless, infinitely thin plane pistons in each end of the aperture, whose motions simulate the movement of the air particles at these positions under acoustic excitation. In the paper (Chen et al., 2006) the improvement on the acoustic transmission loss of a duct by adding some Helmholtz resonator is discussed. Therefore, the calculation on the transmission loss of a duct in a rigid wall by modifying the formula derived by Wilson and Soroka were done.
In paper Sarigug (1999) acoustic surface pressures of various pipes in the shape of a quarter torus were calculated. Sound attenuation spectra of pipes possessing different bend sharpnesses were presented.
A predictive model for sound propagation in tubes with permeable walls was presented in paper Cummings and Kirby (1999). Tubes with permeable walls are used in various applications. The propagation model was verified by comparison to experimental data on a perforated metal tube and was applied to a practical type of permeable fabric tube.
The study of the refection and transmission coefficients of acoustic waves in ducts of continuously varying cross-sectional area has been of interest to many researchers in the past. Various numerical methods for predicting these coefficients have been developed, such as the method of weighted residuals, the finite element method, the perturbation method, the boundary element method, and the matricial Riccati equation method (Utsumi, 2001). A method of solution based on a variational principle has been presented in paper (Utsumi, 2001) for the acoustic wave propagation in ducts having a continuous change in cross-sectional area.
Finite-element methods for time-harmonic acoustics governed by the reduced wave equation (Helmholtz equation) have been an active research area for nearly 40 years. Initial applications of finite-element methods for time-harmonic acoustics focused on interior

problems with complex geometries including direct and modal coupling of structural acoustic systems for forced vibration analysis, frequency response of acoustic enclosures, and waveguides (Thompson, 2006).

There are lot of papers presenting experimental and numerical methods to predict sound transmission loss in silencers. In general, only simple shaped silencers may be analyzed by conventional analytical approach. The silencers of complicated shape with or without acoustic internal structures have been analyzed by transfer matrix method, FEM (Finite Element Method), the transfer matrix method with BEM, multi-domain BEM (Boundary Element Method), multi-domain structural-acoustic coupling analysis, etc (Ju & Lee, 2005). Among these, the transfer matrix method with BEM has been used to calculate the transmission or insertion loss of the silencers. In this method, the particle velocities and sound pressures of each domain of the silencer are calculated by BEM and the four pole parameters of every domain are calculated.

The calculation of attenuation of sound in the acoustic systems with complicated internal structures by the conventional BEM combined with the transfer matrix method is incorrect at best or impossible for 3-dimensional domains due to its inherent plane wave assumption.

On this consideration, in this chapter it is proposed an efficient practical numerical method (based on FEM) for calculation of attenuation of sound within duct/pipe and the whole acoustic structure. The transmission loss estimation by the proposed numerical method is tested by comparison with the experimental one on an sound attenuation in reflective pipe. The method shows its viability by presenting the reasonably consistent anticipation of the experimental results.

2. Sound attenuation

A lot of devices with duct systems like air-condition, heating equipment, frig and so on are being used in people's life. External boundaries can channel sound propagation, and in some cases can create buildup or attenuation of acoustic energy within a confined space. Pipes or ducts acts as guides of acoustic waves, and the net flow of energy, other than that associated with wall dissipation, is along the direction of the duct. The general theory of guided waves applies and leads to a representation in terms of guided modes.

When sound impinges on a surface of external boundaries, some sound is reflected and some is transmitted and possibly absorbed within or on the other side of the surface. To understand the processes that occur, it is often an appropriate idealization to take the incident wave as a plane wave and to consider the surface as flat.

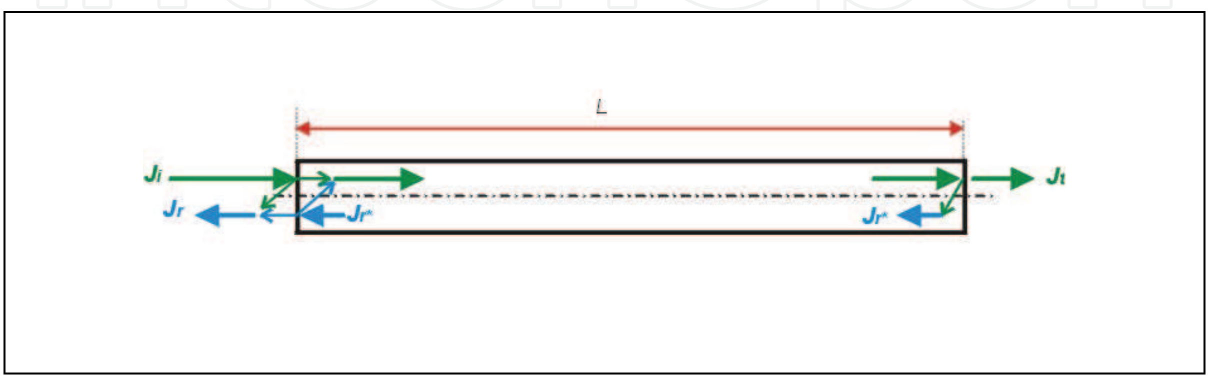


Fig. 1. A description of sound power transmission in the pipe with outlet to free space

When a plane wave reflects at a surface with finite specific acoustic impedance Z_S, a reflected wave is formed such that the angle of incidence Θ equals the angle of reflection (law of mirrors). Here both angles are reckoned from the line normal to the surface and correspond to the directions of the two waves.

In this paper the reflective pipe of length L and radius r with rigid walls is analysed. Fig. 1 shows sound power transmission, where J_i - sound power from source, J_r - sound power reflected from pipe inlet, J_{r^*} - sound power reflected from pipe outlet (if $L \to \infty$ then $J_{r^*} \to 0$) and J_t - sound power transmitted by outlet outside pipe.

The difference between the sound energy on one side of the pipe and that radiated from the second side (both expressed in decibels) is called the sound transmission loss. The larger the sound transmission loss (in decibels), the smaller the amount of sound energy passing through and consequently, less noise heard. Transmission loss depends on frequency.

For the ideal case, when there is no ambient velocity and when viscosity and thermal conduction are neglected, the energy density in considered domain at given time is described by (Rossing, 2007)

$$w = \frac{1}{2}\left(\rho v^2 + \frac{p^2}{\rho c^2}\right). \tag{2}$$

The first term in the above expression for energy density is recognized as the acoustic kinetic energy per unit volume, and the second term is identified as the potential energy per unit volume due to compression of the fluid.

The acoustic intensity is described by

$$\mathbf{I} = p\mathbf{v} \tag{3}$$

where $\mathbf{I}$ is an intensity vector or energy flux vector, p is the pressure and $\mathbf{v}$ is the fluid velocity.

For a plane wave the kinetic and potential energies are the same, and that the energy density is given by

$$w = \frac{p^2}{\rho c^2} \tag{4}$$

and the intensity becomes

$$\mathbf{I} = \mathbf{n}\frac{p^2}{\rho c}. \tag{5}$$

For such a case, the intensity and the energy density are related by

$$\mathbf{I} = c\,\mathbf{n}\,w. \tag{6}$$

This yields the interpretation that the energy in a sound wave is moving in the direction of propagation with the sound speed. Consequently, the sound speed can be regarded as an energy propagation velocity (Rossing, 2007).

The human ear can generally perceive sound pressures over the range from about 20μPa up to about 200 Pa. Because the range of typical acoustic pressures is so large, it is convenient to work with a relative measurement scale rather than an absolute measurement scale. These scales are expressed using logarithms to compress the dynamic range (Rossing, 2007).

One can write the sound intensity level (SIL) as the logarithm of two intensities

$$SIL = 10\,Log\left(\frac{I}{I_{ref}}\right) \tag{7}$$

where I is the intensity of sound wave and I_{ref} is the reference intensity. For the intensity of a sound wave in air, the reference intensity is defined to be $I_{ref} = 10^{-12}\,W/m^2$. Additionally, the sound pressure level (SPL) is defined as

$$SPL = 20Log\left(\frac{p}{p_{ref}}\right) \tag{8}$$

where p is the acoustic pressure and p_{ref} is the reference pressure. For sound in air the reference pressure is defined as 20μPa. Sound levels and parameters of the transmission properties are measured in units of decibels (dB).

There are a few measure parameters of the transmission properties of acoustic devices (i.e. muffler, duct) which are used by engineers. Insertion loss (IL), which is known to be a correct performance measure of a acoustic device, is defined as the difference between the sound powers without and with a device in place. By measuring the sound pressure levels SPL_1 and SPL_2 without and with device, respectively, the insertion loss of the device can be calculated using the following relationship

$$IL = SPL_1 - SPL_2\,. \tag{9}$$

A useful measure of the acoustic performance which depends only on the device and not also on connected elements is the frequency dependent sound transmission loss (Byrne et al., 2006). It is defined as the ratio of the sound power incident on the device inlet to that of the sound power leaving the device at the outlet. Transmission loss (TL) also know as Sound Reduction Index (SRI) is given by

$$TL = 10\,log\frac{W_i}{W_t}\,, \tag{10}$$

where W_i denotes the incoming power at the inlet, W_t denotes the transmitted (outgoing) power at the outlet.

To determine the transmission loss in the above model, one must first calculate the incident and transmitted time-averaged sound intensities and the corresponding sound power values. The incoming and transmitted sound powers are given by equations

$$W_i = \int_A \frac{p_0^2}{2\rho c} dA \ , \ \ W_t = \int_A \frac{|p|^2}{2\rho c} dA \ , \tag{11}$$

where p_0 represents the applied pressure source amplitude and A - the area of boundary where waves are incoming to the pipe and outgoing from the pipe.
Please notice that incoming sound power is expressed by applied pressure source amplitude and not by pressure at the inlet. Because calculated sound power at the inlet is sum of sound power from the source, sound power reflected from pipe inlet, pipe outlet and pipe boundaries. Effective incoming sound power at inlet is difference of sound power from the source and sound power reflected from pipe inlet. In sound transmission loss incoming sound power is expressed by effective incoming sound power.
Most of the energy entering and leaving the device is carried by plane waves. Hence, the sound energy incident on transmitted from the device can be found from the acoustic pressures associated with the incident and transmitted acoustic waves. If the acoustic pressure time histories of an appropriate transient incident wave and the corresponding transmitted wave are Fourier Transformed. The ratio of the moduli of their transforms are taken and expressed in logarithmic form, the required frequency dependent sound transmission loss of the device can be found. That is

$$TL = 20 \log \frac{FFT_i}{FFT_t} \ \ [\mathrm{dB}] \tag{12}$$

where FFT_i and FFT_t are the Fourier Transforms of the time histories of the incident and transmitted waves, respectively.

3. Theory background – mathematical model and computational method

3.1 Linear acoustic

Theory in this section is based on handbook of acoustic (Rossing, 2007). This book is the usual starting reference for learning the details of acoustic. If viscosity and thermal conductivity are neglected at the outset, then the Navier-Stokes equation reduces to the Euler equation

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g} \ . \tag{13}$$

The conservation of mass is described by the partial differentia equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \ . \tag{14}$$

where $\rho = \rho(\mathbf{x}, t)$ is the (possibly position- and time-dependent) mass density and $\mathbf{v}$ is the local and instantaneous particle velocity.
The differential equation of state yields

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} \tag{15}$$

where $\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)$. Equation (14) with combination with equation (15) yields

$$\frac{Dp}{Dt} = -\rho c^2 \nabla \cdot \mathbf{v} \,. \tag{16}$$

The equations that one ordinarily deal with in acoustics are linear in the field amplitudes, where the fields of interest are quantities that depend on position and time.
Sound results from a time-varying perturbation of the dynamic and thermodynamic variables that describe the medium. For sound in fluids (liquids and gases), the quantities appropriate to the ambient medium (i.e. the medium in the absence of a disturbance) are customarily represented by the subscript 0, and the perturbation are represented by a prime on the corresponding symbol.
Thus one expresses the total pressure as

$$p = p_0 + p' \tag{17}$$

with corresponding expressions for fluctuations In specific entropy, fluid velocity and density.
The linearized version of the conservation of mass relation is

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot \left(\mathbf{v}_0 \rho' + \mathbf{v}' \rho_0\right) = 0 \,. \tag{18}$$

Due to restrictions on using linear equations it should be noted that one regards $p' << \rho_0 c^2$, $|\mathbf{v}'| << c$. It is not necessary that p' be much less than p_0, and it is certainly not necessary that $|\mathbf{v}'|$ be less than $|\mathbf{v}_0|$.
The equations for linear acoustics neglect dissipation processes and consequently can be derived from the equations for flow of a compressible ideal fluid. If one neglects gravity at the outset, and assumes the ambient fluid velocity is zero, then the ambient pressure is constant. In such a case (16) leads to

$$\frac{\partial p}{\partial t} + \rho c^2 \nabla \cdot \mathbf{v} = 0 \,. \tag{19}$$

The Euler equation leads to

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \ . \tag{20}$$

Here a common notational convention is used to delete primes and subscripts. The density ρ here is understood to be the ambient density ρ_0, while p and $\mathbf{v}$ are understood to be the acoustically induced perturbations to the pressure and fluid velocity.

A single equation for the acoustic part of the pressure results when one takes time derivative of (19) and then uses (20) to express the time derivative of the fluid velocity in terms of pressure. The resulting equations for the case when the density varies with position is

$$\frac{\partial^2 p}{\partial t^2} = \rho c^2 \nabla \cdot \left(\frac{1}{\rho} \nabla p \right), \tag{21}$$

where $K = \rho c^2$ is called the adiabatic bulk modulus. When density is independent from position we have

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p \ . \tag{22}$$

An important special case is a time-harmonic wave, for which the pressure varies with time as

$$p(\mathbf{x}, t) = p(\mathbf{x}) e^{i\omega t} \tag{23}$$

where $\omega = 2\pi f$ is the angular frequency, with f denoting the frequency. Assuming the same harmonic time dependence for the source terms, the wave equation for acoustic waves reduces to an inhomogeneous Helmholtz equation

$$-\frac{\omega^2 p}{\rho \ c^2} = \nabla \cdot \left(\frac{1}{\rho} \nabla p \right). \tag{24}$$

For the acoustic pressure in 2D axisymmetric geometries the wave equation becomes

$$\frac{\partial}{\partial r}\left(-\frac{r}{\rho}\frac{\partial p}{\partial r} \right) + r\frac{\partial}{\partial r}\left(-\frac{1}{\rho}\frac{\partial p}{\partial z} \right) - \left[\left(\frac{\omega}{c} \right)^2 - \left(\frac{m}{r} \right)^2 \right] \frac{rp}{\rho} = 0 \tag{25}$$

where m denotes the circumferential wave number and k_z is the out-of-plane wave number. In 2D axisymmetric geometries the independent variables are the radial coordinate r and the axial coordinate z.

Typical boundary conditions are: sound-hard boundaries (walls), sound-soft boundaries (zero acoustic pressure), specified acoustic pressure, specified normal acceleration, impedance boundary conditions and radiation boundary conditions (Comsol, 2007).

3.2 The dimensionless equations

For simplicity the preferred work choice is to work in non-dimensional frame of reference. Now some dimensionless variables will be introduced in order to make the system much easier to study. This procedure is very important so that one can see which combination of parameters is more important that the others.

Non-dimensionalised variables and scales are defined as follow: time $t' = \frac{c}{h} t$; distance variable $\mathbf{x}' = \frac{1}{h}\mathbf{x}$; pressure $p' = \frac{p}{\rho \mathrm{c}^2}$; angular frequency $\omega' = \frac{h}{c}\omega$; monopole source $Q' = \frac{Q}{Q_0}$ are introduced, where h is characteristic distance (for instance pipe length).

The next step is to put these variables into the wave equation (Comsol, 2007)

$$\frac{1}{\rho c^2}\frac{\partial^2 p}{\partial t^2} + \nabla \cdot \left(-\frac{1}{\rho}\nabla p \right) = Q \, , \tag{26}$$

what gives

$$\frac{1}{\rho c^2}\frac{\rho c^2}{(h/c)^2}\frac{\partial'^2 p'}{\partial' t'^2} + \frac{1}{h}\nabla' \cdot \left(-\frac{1}{\rho}\frac{\rho c^2}{h}\nabla' p' \right) = Q_0 Q \, , \tag{27}$$

or after mathematical manipulations with $\rho = const$

$$\left(\frac{c}{h}\right)^2 \frac{\partial'^2 p'}{\partial' t'^2} + \left(\frac{c}{h}\right)^2 \nabla'^2 p' = Q_0 Q \tag{28}$$

where Q [$1/s^2$] monopoly source.

Then multiply through by $\frac{h^2}{c^2}$ what gives

$$\frac{\partial'^2 p'}{\partial' t'^2} + \nabla'^2 p' = \frac{h^2 Q_0}{c^2} Q' \, . \tag{29}$$

Putting dimensionless variables into the time-harmonic wave equation we get

$$-\omega'^2 p' - \nabla'^2 p' = 0 \, , \tag{30}$$

and into the 2D axisymmetric time-harmonic wave equation we get

$$\frac{\partial'}{\partial' r'}\left(-r'\frac{\partial' p'}{\partial' r'}\right)+r'\frac{\partial'}{\partial' r'}\left(-\frac{\partial' p'}{\partial' z'}\right)-\left[\omega'^2-\left(\frac{m'}{r'}\right)^2\right]\frac{r' p'}{\rho}=0\ . \tag{31}$$

Since now primes will not be written (old variables symbols will be used) but it's important to remember that they are still there.

3.3 The basic of Finite Element Method

Many CFD practitioners prefer finite volume methods because the derivation of the discrete equations is based directly on the underlying physical principles, thus resulting in "physically sound" schemes. From a mathematical point of view, finite volume, difference, and element methods are closely related, and it is difficult to decide that one approach is superior to the others; these spatial discretization methods have different advantages and disadvantages.

Today the Finite Element Method (FEM) has been widely employed in solving field problems arising in modern industrial practices (Zienkiewicz & Taylor, 2000; Hinton & Owen, 1979; Huang et al., 1999; Huebner, 1975). The text in this section is short introduction to the basis of the FEM to the analysis of acoustic problems which are a very common phenomenon in many processes of engineering.

Some physical problems can be stated directly in the frame of variational principle which consists of determining the function which makes a certain integral statement called functional stationary. However the form of the variational principle is not always obvious and such a principle does not exist for many continuum problems.

As an alternative to solve such differential equations we may use a variety of weighted residual methods. Weighted residual methods are numerical techniques which can be used to solve a single or set of partial differential equations. Consider such a set in domain Ω with boundary $\delta\Omega=\Gamma$, where u is the exact solution and may represent a single variable or a column vector of variables.

Applying the method of weighted residuals involves basically two steps. The first step is to assume the general functional behaviour of the dependent field variable in some way so as to approximately satisfy the given differential equation and boundary conditions. Substitution of this approximation into the original differential equation and boundary conditions then results in some error called a residual. This residual is required to vanish in some average sense over the solution domain. The second step is to solve the equation (or equations) resulting from the first step and thereby specialize the general functional form to a particular function, which then becomes the approximate solution sought. According to Galerkin's method, the weighting functions are chosen to be the same as the approximating functions.

Let us consider time independent differential equation

$$L(u)=f \tag{32}$$

defined within a domain Ω and with boundary conditions specified at the boundary of Γ. $L(u)$ typically includes derivatives of u up to second order.
In the weighted-residual method the solution u is approximated by expressions $\overline{u}$ of the form

$$\overline{u} = S_0 + \sum_{j=1}^{N} u_j S_j \tag{33}$$

where S_j are trial functions, and S_0 must satisfy all the specified boundary conditions ($S_0 = 0$ if all the specified boundary conditions are homogeneous) of the problem, and S_i must satisfy the following conditions:

- S_j should be such that $L(S_j)$ is well defined and nonzero, i.e. sufficiently differentiable;
- S_j must satisfy at least the homogeneous form of the essential boundary conditions of the problem;
- for any N, the set $\{S_j, j = 1,2,...N\}$ is linearly independent.

We begin by introducing the error, or residual, R_Ω in the approximation (by substitution of the approximation $\overline{u}$ into the operator equation) which is defined by

$$R_\Omega = L(\overline{u}) - f \tag{34}$$

where $\overline{u}$ contains trial functions and satisfies the Dirichlet boundary conditions of $\overline{u} = u_0$ at $\Gamma_1 \subseteq \Gamma$. If the residual is smaller the approximation is better. It should be noted that R_Ω is a function of position in Ω. Now we attempt to reduce this residual as close to zero as possible. If we have

$$\int_\Omega T_i R_\Omega d\Omega = 0 \tag{35}$$

where $T_i, \mathrm{i} = 1,2,...,\mathrm{M}$ is a set of arbitrary functions and $M \to \infty$, then it can be said that the residual R_Ω vanishes. Here T_i are called weighting functions which, in general, are not the same as the approximation (trial) functions S_i. Expanding above equation we have

$$\int_\Omega T_i \left(L(\overline{u}) - f\right) d\Omega = 0\ . \tag{36}$$

A function $\overline{u}$ that satisfies above equation for every function T_i in Ω is a weak solution of the differential equation, whereas the strong solution $\overline{u}$ satisfies the differential equation at every point of Ω.
When the operator L is linear above equation can be simplified to the form

$$\sum_{j=1}^{N}\left(\int_{\Omega} T_i L(S_j) d\Omega\right) u_j = \int_{\Omega} T_i (f - L(S_0)) d\Omega \tag{37}$$

or

$$\sum_{j=1}^{N} A_{ij} u_j = f_i \tag{38}$$

where

$$A_{ij} = \int_{\Omega} T_i L(S_j) d\Omega \tag{39}$$

and

$$f_i = \int_{\Omega} T_i (f - L(S_0)) d\Omega . \tag{40}$$

Note that the coefficients of matrix $\mathbf{A}$ is not symmetric $A_{ij} \neq A_{ji}$.

The weighted-residual method (when $T_i \neq S_i$) is also sometimes referred to as the Petrov-Galerkin method. For different choices of T_i the method is known by different names. We outline below the most frequently used methods.

The Galerkin method. For $T_i = S_i$ the weighted-residual method is known as the Galerkin method. When the operator is a linear differential operator of even order, the Galerkin method reduces to the Ritz method. In this case the resulting matrix will be symmetric because half of the differentiation can be transformed to the weight functions.

The least-squares method. The least-squares methods determines the constants u_j by minimizing the integral of the square of the residual

$$\frac{\partial}{\partial u_i} \int_{\Omega} R_{\Omega}^2 d\Omega = 0 \tag{41}$$

or

$$\int_{\Omega} \frac{\partial R_{\Omega}}{\partial u_i} R_{\Omega} d\Omega = 0 \tag{42}$$

A comparison of Eq. (42) with Eq. (35) shows that $T_i = \frac{\partial R_{\Omega}}{\partial u_i}$. If L is a linear operator Eq. (37) becomes

$$\sum_{j=1}^{N}\left(\int_{\Omega} L(S_i) L(S_j) d\Omega\right) u_j = \int_{\Omega} L(S_i)(f - L(S_0)) d\Omega \tag{43}$$

which yields a symmetric matrix but requires the same order of differentiation as the operator equation.
The collocation method. The collocation method seeks approximate solution $\bar{u}$ by requiring the residual $R_\Omega = R_\Omega(\mathbf{x},\mathbf{u})$ in the equation to be identically to zero at N selected points $\mathbf{x}_i$, $i = 1,2,...,N$ in the domain Ω

$$R_\Omega(\mathbf{x}_i, u_j) = 0 . \tag{44}$$

The selection of the points $\mathbf{x}_i$ is crucial in obtaining a well conditioned system of equations and ultimately in obtaining an accurate solution. The collocation points can be shown to be a special case of Eq. (34) $\int_\Omega T_i R_\Omega d\Omega = 0$ for $T_i = \delta(\mathbf{x} - \mathbf{x}_i)$, where $\delta(\mathbf{x})$ is the Dirac delta function

$$\int_\Omega f(\mathbf{x})\delta(\mathbf{x} - \xi)d\Omega = f(\xi) . \tag{45}$$

The Courant method. To so-called Courant method combines the basic concepts of the Ritz method and the least-squares method (for linear operator). The method seeks approximate solution $\bar{u}$ by minimizing the modified quadratic functional

$$I_p(\bar{u}) = I(\bar{u}) + \frac{\alpha}{2}\left\| L(\bar{u}) - f \right\|^2 \tag{46}$$

where $I(u)$ is the quadratic functional associated with $L(u) = f$, when L is linear, and α is the penalty parameter. Obviously the statement make sense only for operator equation that admit functional formulation.

4. Numerical results

In this section the time-harmonic wave analysis of idealized reflective pipe with rigid walls and outlet to free acoustic space is made. The frequency dependent sound transmission loss of hard-walled pipe is calculated as well. The pipe length is L and the radius is r. Free acoustic space beyond pipe outlet (no reflected wave is coming back to pipe outlet) is modelled by larger pipe with impedance boundary condition.
One can use Perfectly Matched Layers (PMLs) to model space outside the outlet pipe (Comsol, 2007). The PMLs serve to absorb the outgoing waves so that the non-physical reflections at their exterior boundaries have a minimal influence on the pressure field inside the pipe.
Fig.2A shows example of non-dimensional pipe with length $L = 1$, radius $r = 0.2$ and free acoustic space modelled by cylinder with $L = 2$ and radius $r = 2$. The axial symmetry of the geometry and the physics makes it natural to set the model up in a 2D axisymmetric application mode (see Fig.2B).

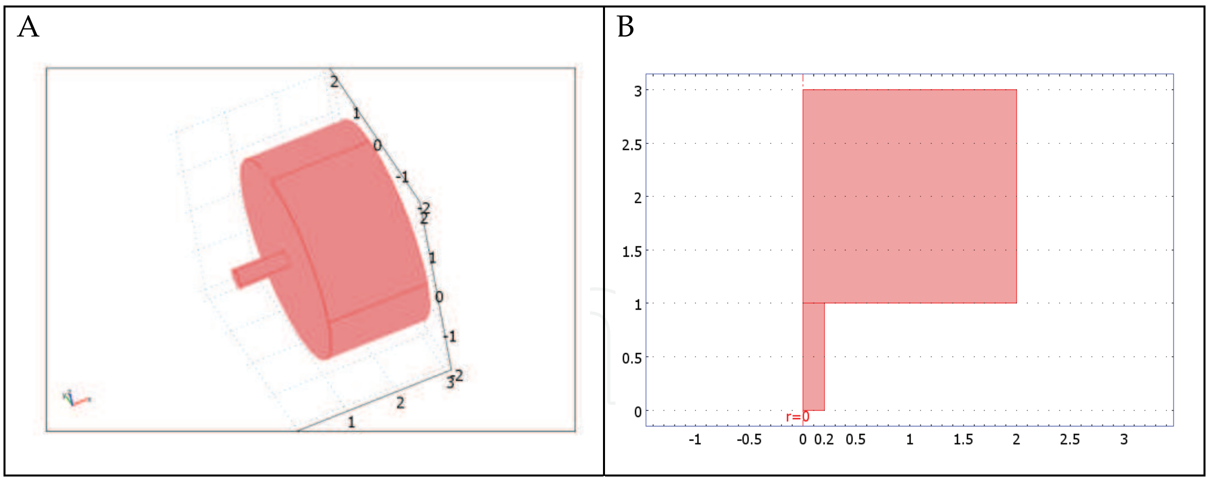


Fig. 2. Geometry: (A) three-dimensional domain and (B) 2D axisymmetric mode

All acoustic problems considered in this work are governed by dimensionless equations with appropriate boundary and initial conditions. Numerical results are obtained using standard computational code COMSOL Multiphysics (Comsol, 2007). Finite element calculations are made with second-order triangular Lagrange elements.
First acoustic pressure is calculated using dimensionless time-harmonic wave equation (31) in considered axisymmetric domain.
Following boundary conditions for dimensionless model are assumed:

- pipe inlet: spherical wave is assumed;
- pipe wall: sound hard boundary $\mathbf{n}\cdot(-\nabla p)=0$;
- cylinder's wall: impedance boundary condition $n\cdot(-\nabla p)=\frac{i\omega p}{Z}$ where Z is impedance and $Z=1$ in dimensionless form;
- axial symmetry $r=0$.

Next using

$$W_i=\int_A p_0^2 dA,\quad W_t=\int_A |p|^2 dA \tag{47}$$

the incoming and transmitted dimensionless sound powers are calculated, respectively. Using calculated sound powers transmission loss (TL) is derived using Eq. (10).
To receive dimension frequency of time-harmonic wave, one can multiply dimensionless frequency by $\frac{c}{L}$ using formula $f=\frac{c}{L}f'$, where f' is dimensionless frequency, L is length of pipe and c is sound velocity in medium. All figures in this chapter with frequency dependent sound transmission loss diagram present TL on vertical axis and dimensionless frequency on horizontal axis. Firstly, the influence of boundary condition at inlet on sound transmission loss is shown at Fig. 3. Two boundary conditions are assumed for pipe with ratio $r/L=0.1$: plane wave with the applied pressure source amplitude p_0 and pressure

$p = p_0$ with $p_0 = 1$. One can observe a big difference between frequency dependent sound transmission loss values calculated with both conditions.

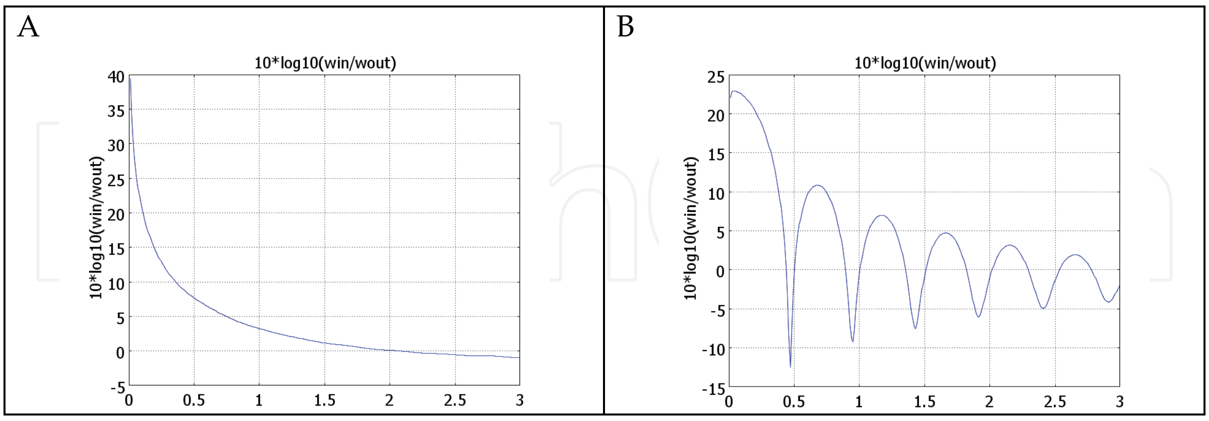


Fig. 3. Influence of inlet boundary condition on TL of pipe with ratio $r/L = 0.1$ versus dimensionless frequency: (A) plane wave, (B) pressure

Secondly, the influence of mesh density in considered domain on sound transmission loss is shown at Fig. 4, as well. The frequency dependent sound transmission loss is calculated for pipe with ratio radius to length equals 0.2 with mesh statistic collected in Table 1. However, FEM has some deficiencies at higher frequencies because of the shorter wavelengths and the higher modal density (see Fig.5). Due to the shorter wavelengths, the number of elements must be increased with frequency.

Number of:	Mesh A	Mesh B
degrees of freedom	951	14601
mesh points	251	3701
elements	450	7200

Table 1. Mesh statistic for pipe with ratio $r/L = 0.2$

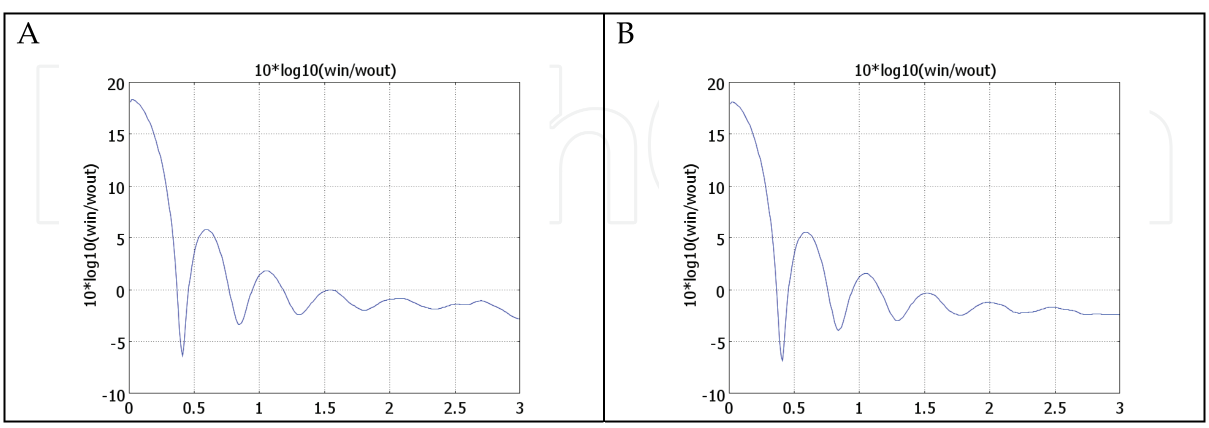


Fig. 4. Influence of mesh density on TL of pipe (ratio $r/L = 0.2$): (A) mesh A, (B) mesh B

Next computational results at Fig. 6. show very important influence of outlet boundary condition on the frequency dependent sound transmission loss. In proposed method free acoustic space outside pipe outlet is modelled by larger cylinder with impedance boundary condition on all outer boundaries. The radius of this cylinder is at least ten times greater than pipe radius. When cylinder's radius is smaller calculated numerical results are not correct (see Fig.6). In many papers transmitted sound power is calculated on the end of device where impedance boundary condition or plane wave with zero pressure source amplitude is assumed. This assumption does not guarantee proper model of environment outside of pipe outlet.

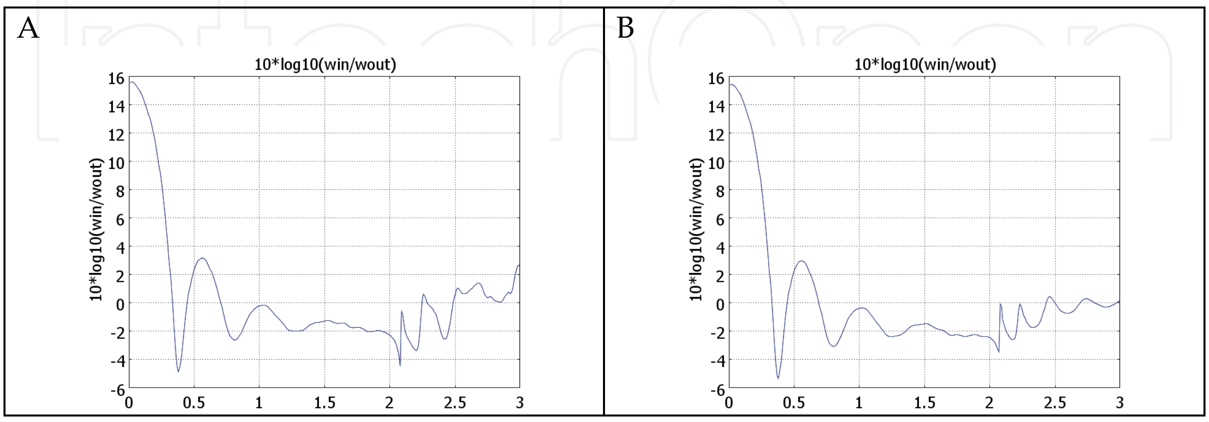
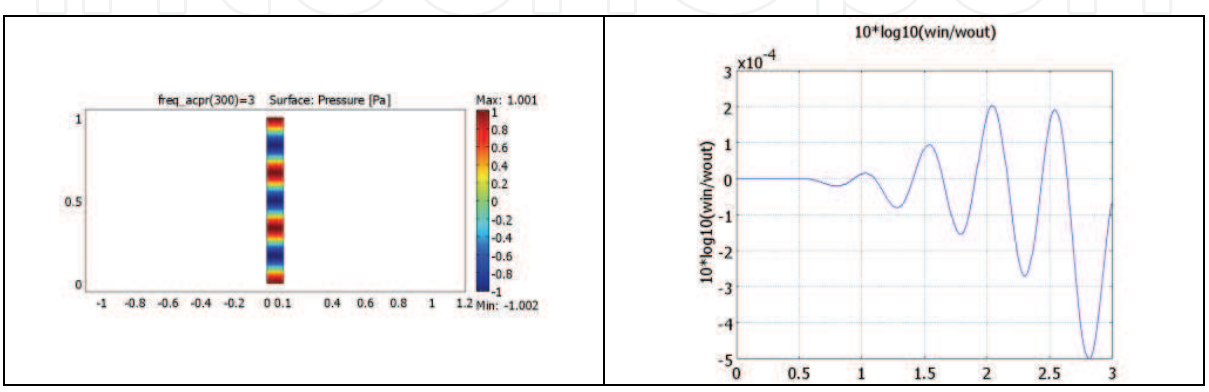


Fig. 5. Influence of mesh density on TL of pipe (ratio $r/L = 0.3$): (A) coarse, (B) fine mesh

Figures 7-9 show comparison of numerical results of calculated frequency dependent sound transmission loss, transmitted sound power and pressure at frequency $f = 3$ for pipes with different radius to length ratio equals 0.015625, 0.0625, 0.1 and 0.2. It is easy to observe that pipe with smaller ratio has greater possibilities to attenuate sound. Absolute value of transmission loss is greater.
Fig. 10 presents numerical results for pipe with ratio $r/L = 0.15$. Left figure presents frequency dependent pressure in point at inlet and outlet of pipe. Both points are placed on axial symmetry line. Right figure presents frequency dependent sound transmission loss for this pipe.
Fig.11 presents comparison of measured transmission loss of 37.5 mm diameter pipe of different lengths with numerical results calculated using 2D axisymmetric non-dimensional model with proper r/L ratio. Measured transmission loss of pipe was found from the Fourier Transforms of the acoustic pressure time histories using Eq. (12). The comparison diagram shows that results of numerical calculations (solid line) are quite good with comparison to experimental data (dashed line) at frequencies smaller than 3.



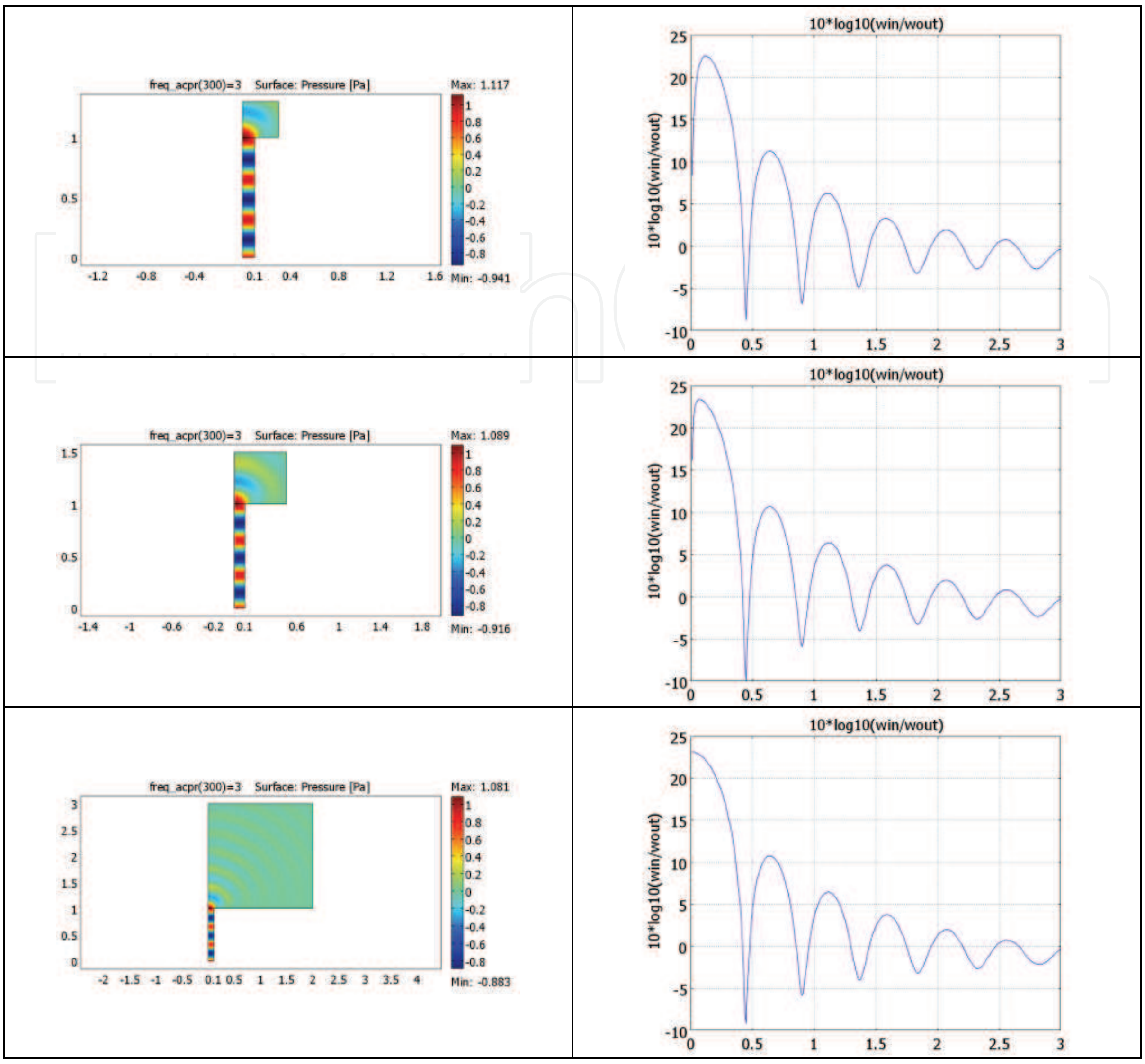
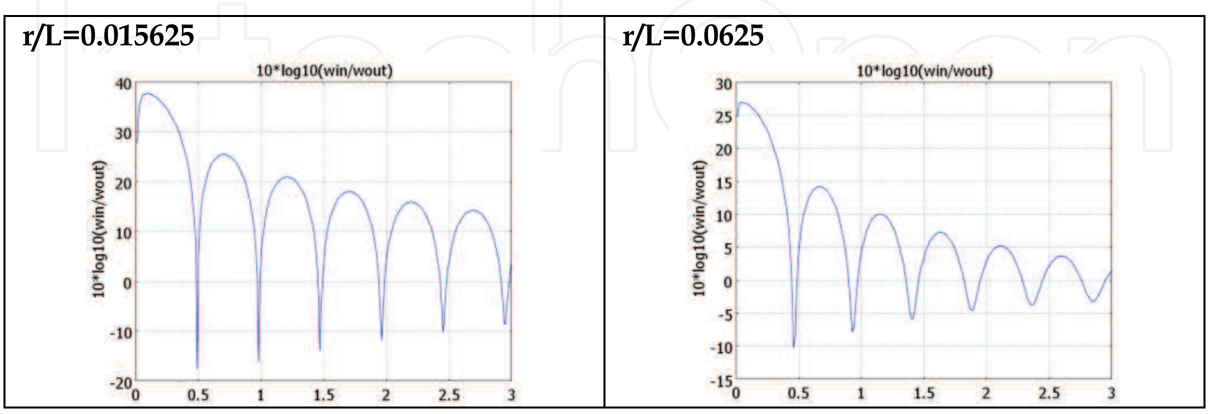


Fig. 6. Results of numerical calculation of transmission loss of pipe (length $L=1$, radius $r=0.1$) with free acoustic space modelled by different size cylinder.



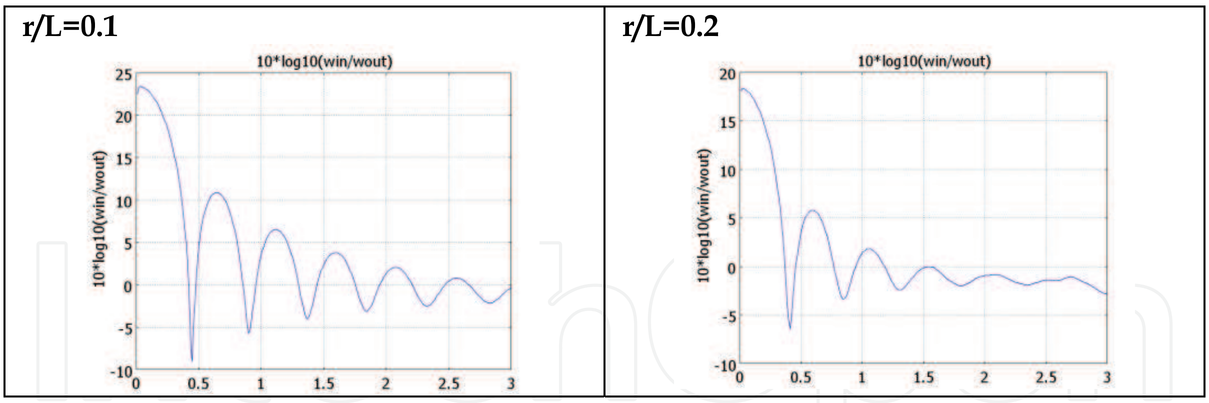


Fig. 7. Comparison of results of numerical calculation of transmission loss with different r/L ratios

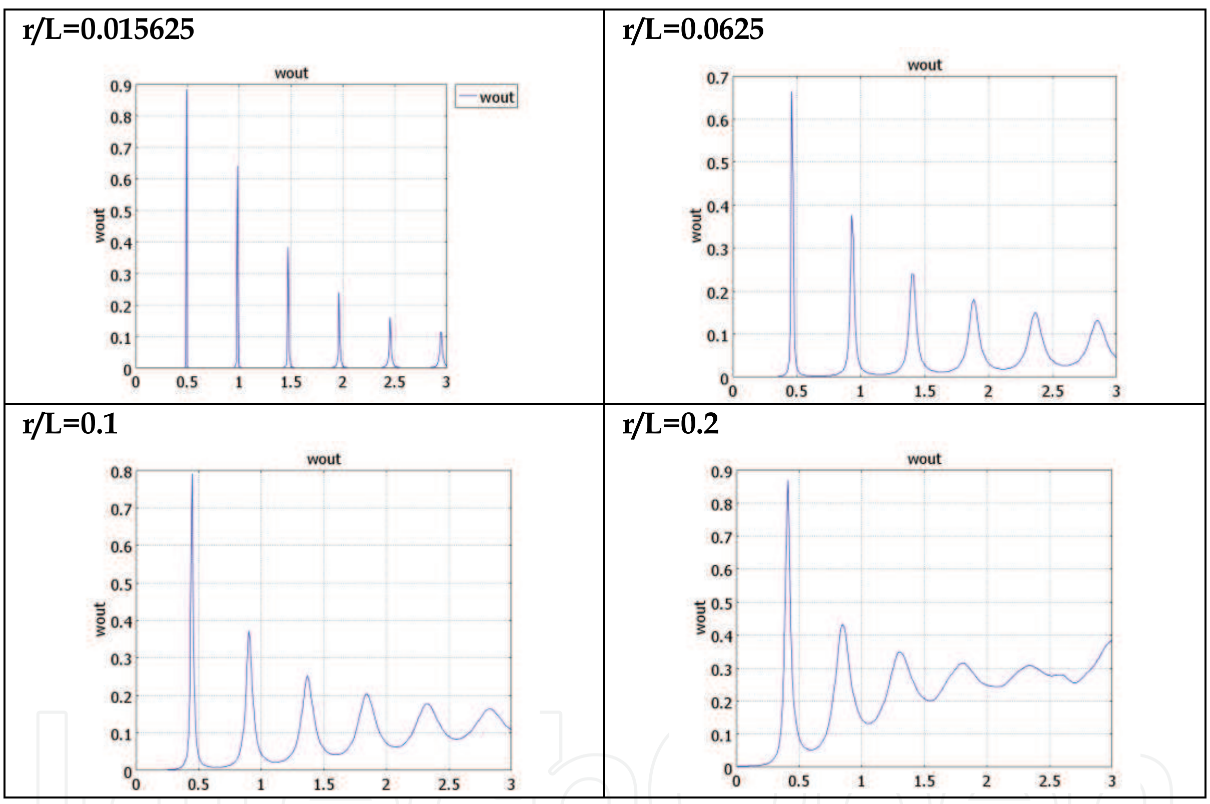


Fig. 8. Comparison of results of numerical calculation of transmitted sound power for pipes with different r/L ratios

Calculated numerical results of the frequency dependent sound transmission loss of reflective pipe for various ratios (radius to length) are compared with measured sound transmission loss for pipes with radius and length collected in Table 2.

Maximum dimension frequency presented on horizontal diagram axis is calculated using dimensionless frequency and formula $f = \frac{c}{L} f'$, where $f' = 3$ is dimensionless frequency, L is length of pipe and $c = 343 m/s$ is sound velocity in medium.

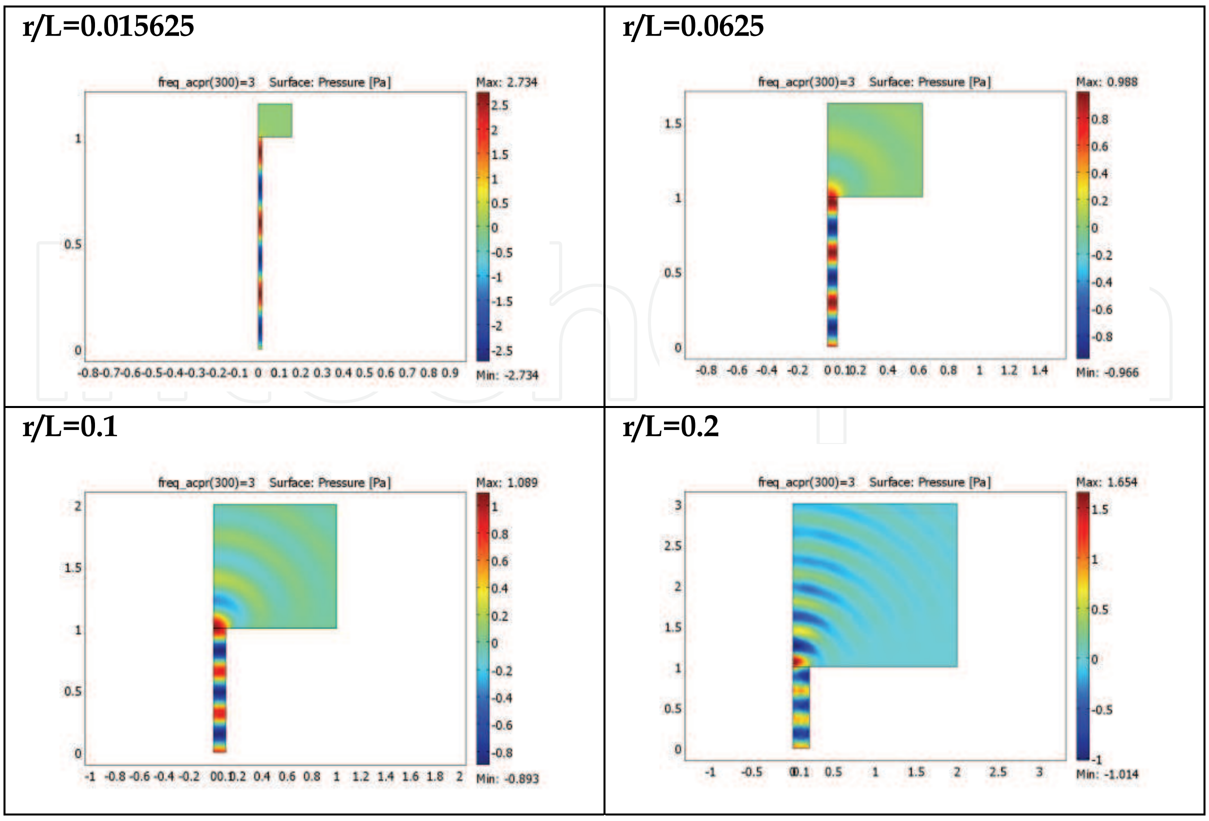


Fig. 9. Comparison of results of numerical calculation of pressure at dimensionless frequency $f = 3$ for pipes with different r/L ratios

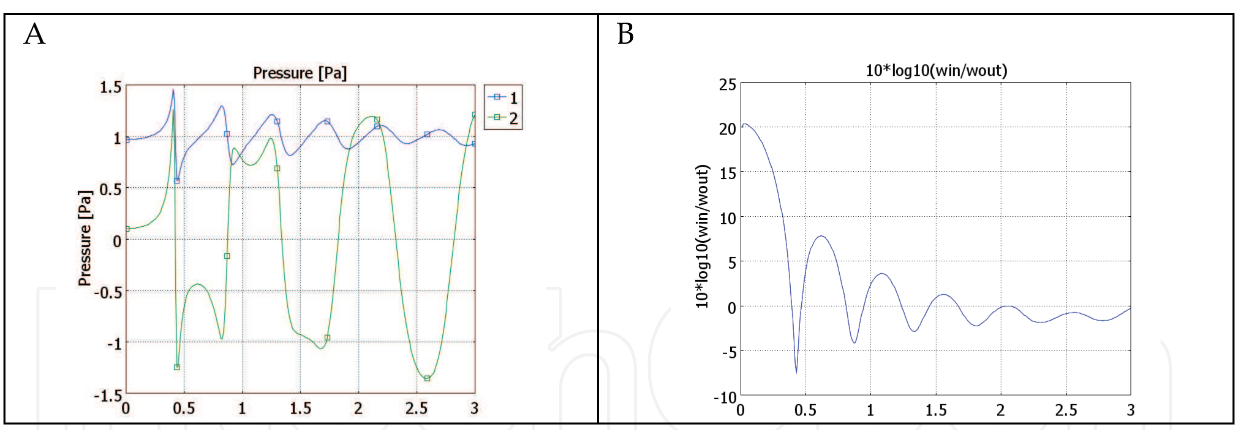


Fig. 10. Numerical results for pipe with $r/L = 0.15$: (A) frequency dependent pressure in point at pipe inlet (line - 1) and outlet (line - 2), (B) transmission loss

There exists for each pipe critical frequency after which calculated and measured frequency dependent sound transmission loss of reflective pipe begin change with different way (see Fig. 12). For example, for pipe with ratio $r/L = 0.015625$ this critical frequency is about 17 and 2 for pipe with ratio $r/L = 0.3$. If frequencies grow then wavelengths diminish and become smaller than size of finite element. Moreover, for high frequencies wave inside hard-walled pipe occurs superposition of longitudinal and transverse waves. These conclusions suggest that computational method is valid for only a limited range of frequencies or wavelengths.

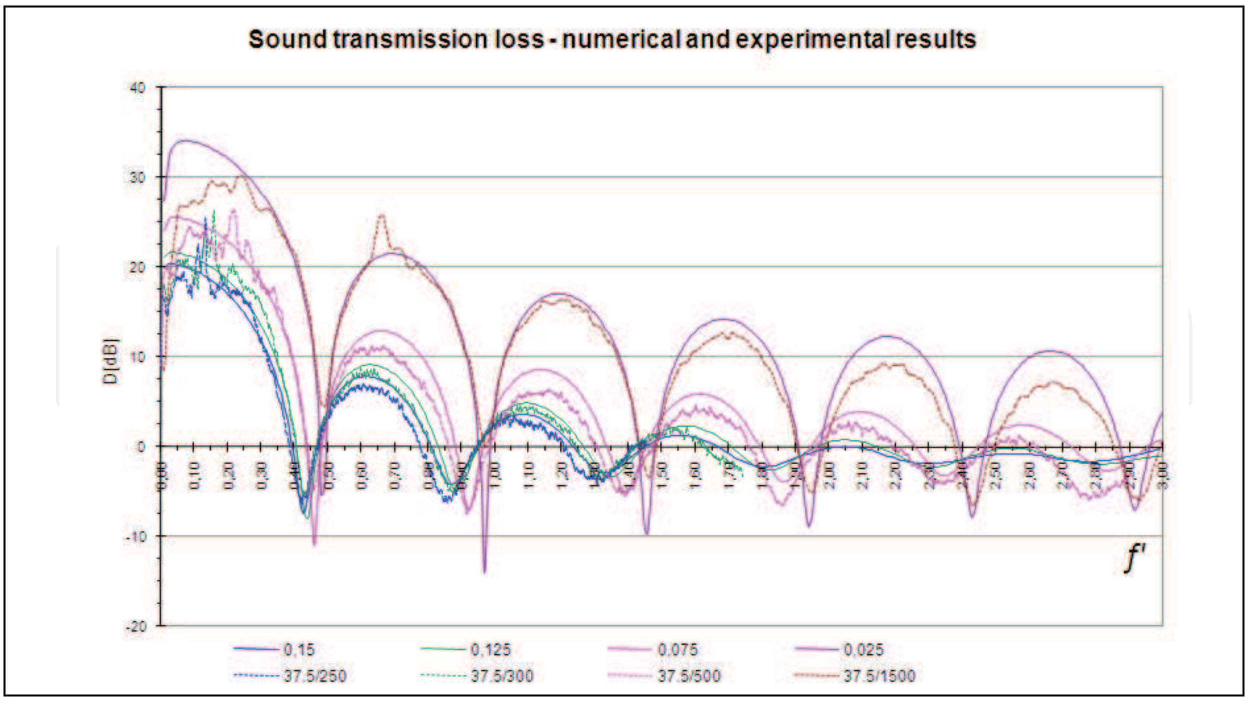


Fig. 11. Comparison of results of numerical calculation of transmission loss with measured transmission loss

Ratio, r/L	Radius, $r\,[mm]$	Length, $L\,[mm]$	Max frequency, $f_{\max}\,[1/s]$
0.025	37.5	1500	686
0.075	37.5	500	2058
0.125	37.5	300	3430
0.15	37.5	250	4116

Table 2. Dimensionless ratio and dimension quantities for pipes

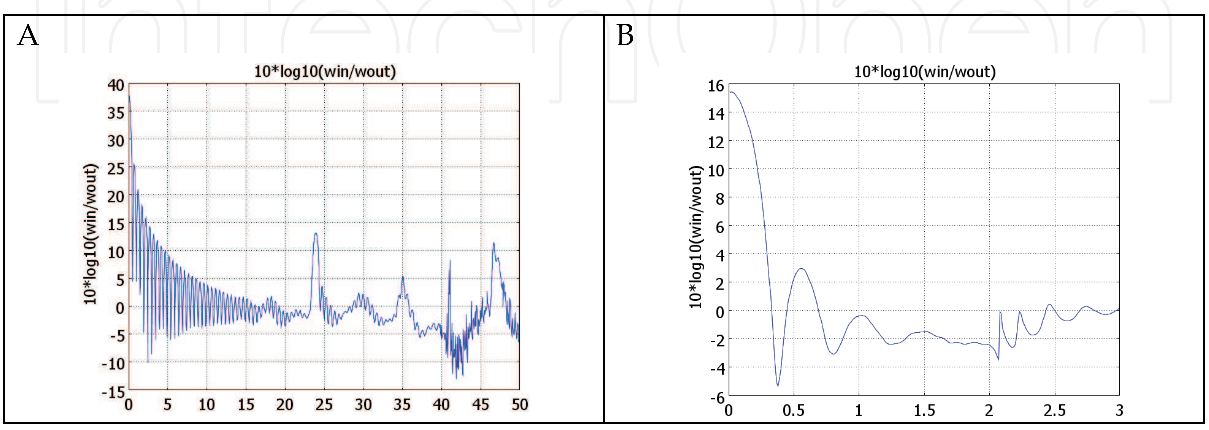


Fig. 12. Frequency dependent sound transmission loss of reflective pipe with ratio: (A) $r/L = 0.015625$, (B) $r/L = 0.3$

5. Conclusions

A lot of tools are available for the study of the time-harmonic wave analysis of acoustic devices. The classical Finite Element Method (FEM) and Boundary Element Method (BEM) are the most conventional tools and are widely used in the low frequency range.

However theoretically correct, these methods have some deficiencies at higher frequencies because of the shorter wavelengths and the higher modal density. In FEM, because of the shorter wavelengths, the number of elements must be increased with frequency. This makes the method costly at high frequencies.

In this chapter numerical method for calculation of the frequency dependent sound transmission loss witin reflective pipe with outlet to free acoustic space is presented. The frequency dependent sound transmission loss of hard-walled pipe with given ratio radius to length is calculated as well.

All acoustic problems considered in this work are governed by dimensionless equations with appropriate boundary and initial conditions. Numerical results are obtained using standard computational code COMSOL Multiphysics using FEM.

The results of this study (for pipes with ratio r/L less than 0.3) show that results of numerical calculations are quite good with comparison to experimental data at dimensionless frequencies less than 3, only if size of cylinder at the pipe outlet is big enough. The FEM results are verified by a calculation with a very fine mesh. The FEM results calculated with a more coarse mesh, show a good agreement with the experiment case.

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Finite element analysis is an engineering method for the numerical analysis of complex structures. This book provides a bird's eye view on this very broad matter through 27 original and innovative research studies exhibiting various investigation directions. Through its chapters the reader will have access to works related to Biomedical Engineering, Materials Engineering, Process Analysis and Civil Engineering. The text is addressed not only to researchers, but also to professional engineers, engineering lecturers and students seeking to gain a better understanding of where Finite Element Analysis stands today.