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Fractional bioeconomic systems: optimal control problems, theory and applications

Darya V. Filatova[a,b], Marek Grzywaczewski[a,c] and Nikolai P. Osmolovskii[c,d,e,f]
[a]Analytical Center RAS
Russian Federation
[b]Jan Kochanowski University in Kielce
Poland
[c]Politechnika Radomska
26-600 Radom, ul. Malczewskiego 20A, Poland
[d]Systems Research Institute,
Polish Academy of Sciences,
ul. Newelska 6, 01-447, Warszawa, Poland
[e]Akademia Podlaska,
ul. 3 Maja 54, 08-110 Siedlce, Poland
[f]This author was supported by grants
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1. Introduction

Exploitation of renewable resources is a task on a global scale inasmuch ecosystems are permanently destroyed by large-scale industrialization and unlimited human population growth. These have made already quit an impact on environment causing climatic destabilization. Thus, prediction of sustainable economic development has to take into account the bioeconomic principles. Although the task is not a new one there is a room for further investigations. It can be explained in the following manner.

It is known that biological systems react on the changes of existence conditions, environment actions and own states. Some of these systems are often utilized in forestry or fishery and therefore human control factor plays a very important role. In order to keep the completeness under uncertain environmental variability and internal transformations the considered biological systems must be in some dynamic equilibrium, which is defined by maximum sustainable yield approach, as a guarantee of the entire system existence. This idea requires removable resource management solved in some optimal sense.

Before the formulation of optimal control problem it is reasonable to notice that despite its popularity maximum sustainable yield (MSY) approach has some obstacles (Clark, 1989). Firstly, it is very sensitive to small errors in population data. Secondly, it does not take into account most of economic aspects of resource exploitation and, at last, it can be hardly used in "species in interaction" cases. It is clear that the problem solution is strongly connected with a task of appropriate mathematical model selection (Jerry & Raissi, 2005; McDonald et al., 2002).

Initially bioeconomic models contained two main components: one defined dynamics of biological system and second characterized the economic policy of selected system exploitation (Clark, 1989). To make them more realistic different types of uncertainties have been incorporated. It was shown that three sources of uncertainty play an important role in fisheries management: variability in fish dynamics, inaccurate stock size estimates, and inaccurate implementation of harvest quotas, but there is not a unique way of how to include noises in models. To describe environmental noise one can use the following principles (Sethi et al., 2005):

- the variance is proportional to the expected population in the next generation;
- environmental fluctuations affect the population multiplicatively (this holds under a range of conditions - the density-independent or maximum growth rate of individuals are affected);
- demographic and environmental fluctuations can have long-range and/or short-range consequences on biological system.

The goal of this work is to show the ways of problem optimal solution when control object meets the principles mentioned above. The rest of the chapter is organized as follows. In Section 2, we formulate the optimal control problem for given tasks, showing how to convert the stochastic task into non-stochastic one. In Section 3, we derive necessary optimality conditions for short-range and long-range dependences, as it requires the object equation, under certain control and state constraints. Finally, Section 4 provides an application of obtained theoretical results to the problem of maximization of expected utility from the terminal wealth.

2. Fractional bioeconomic systems

2.1 A fishery management model

A renewable resource stock dynamics (or population growth) can be given as growth model of the type

$$dX(t) = s(t, X(t))dt, \tag{1}$$

with given initial condition $X(t_0) = X_0$. Here $X(t) \geq 0$ is the size of population at time t, $s(t, X(t))$ is the function, which describes population growth.

Model selection depends on the purpose of the modeling, characteristics of biological model and observed data (Drechsler et al., 2007). Usually one takes

$$s(t, X(t)) = \theta_1 X(t)$$

or

$$s(t, X(t)) = \theta_1 X(t)(1 - \theta_2 X(t)), \tag{2}$$

where $\theta_1 > 0$ is the intrinsic growth rate, $\frac{1}{\theta_2} > 0$ is the carrying capacity, $\theta_2 > 0$.
If in the first case population has an unlimited growth, in the second case we can also show that the population biomass $X(t)$ increases whenever $X(t) < \frac{1}{\theta_2}$, decreases for $X(t) > \frac{1}{\theta_2}$ and is in a state of equilibrium if $X(t) \to \frac{1}{\theta_2}$ as $t \to \infty$ (see Fig. 1).

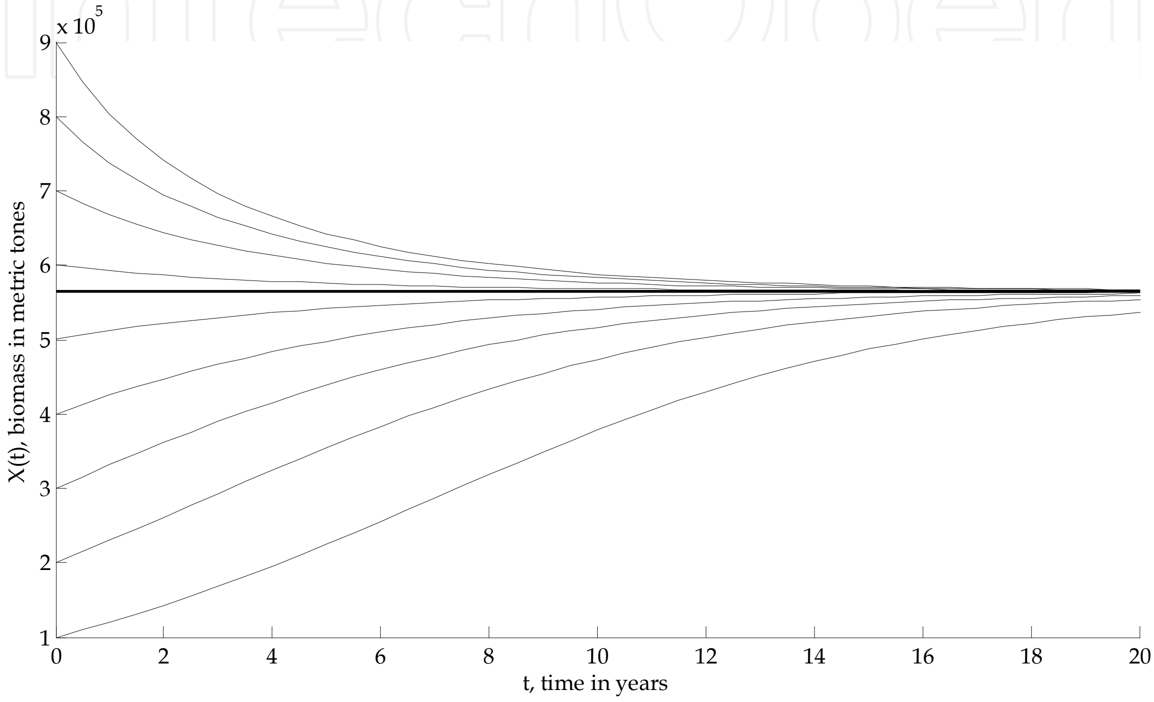

Fig. 1. Changes in population size $X(t)$ predicted by logistic growth function (2) for the southern bluefin tuna (McDonald et al., 2002)

Taking into account continuous harvesting at variable rate $u(t)$ the model (1) can be rewritten as

$$dX(t) = \left[s(t, X(t)) - u(t)\right]dt, \tag{3}$$

where the harvest rate has to be limited, for example

$$0 \le u(t) \le u_{\max}, \tag{4}$$

in order to guarantees the existence of the ecosystem under environmental variability and internal transformations (Edelstein-Keshet, 2005).

Assume that $u(t) = \text{constant}$. In this case the dynamic equation (3) gives a picture of the logistic growth model behavior. So, for $u < \max\left[s\left(t, X(t)\right)\right]$, the equation has one stable (point B on Fig. 2) and one unstable equilibrium (point A on Fig. 2). For $u > \max\left[s\left(t, X(t)\right)\right]$, there is not any equilibrium state. If $u = \max\left[s\left(t, X(t)\right)\right]$, the equation has only a single semistable equilibrium at the point called maximum sustainable yield (point C on Fig. 2). MSY is widely used for finding optimal rates of harvest, however and as it was mentioned before, there are problems with MSY approach (Kugarajh et al., 2006; Kulmala et al., 2008).

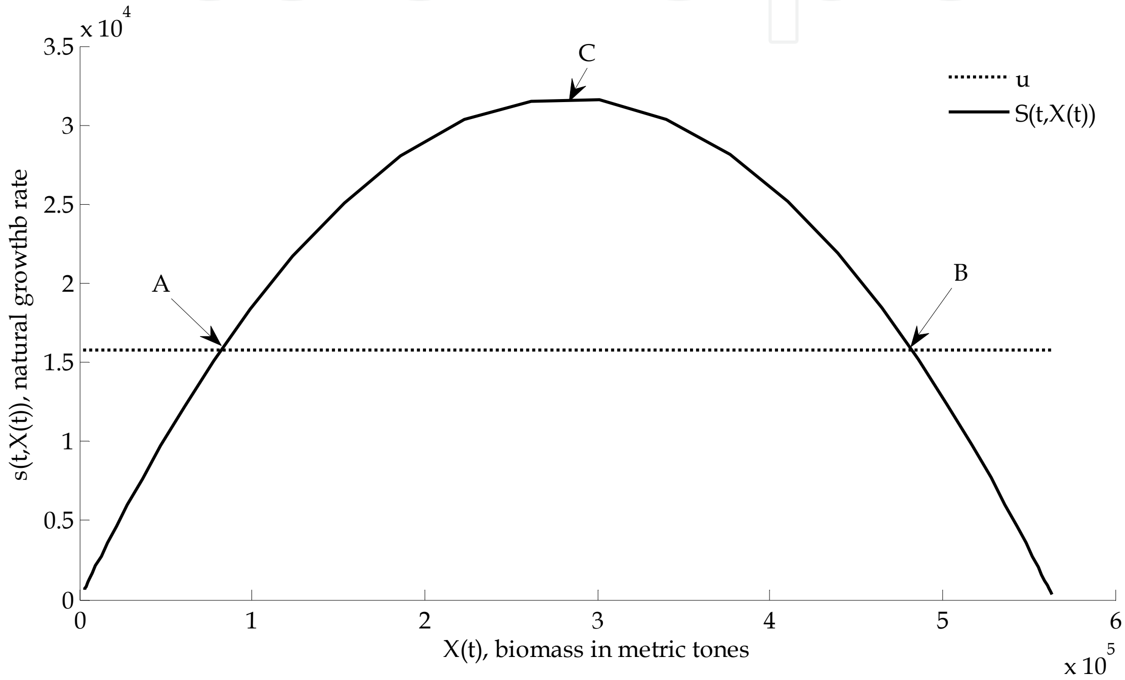

Fig. 2. Population dynamics with constant rate harvesting u for the southern bluefin tuna (McDonald et al., 2002)

To make the model more realistic one has to take into account different types of uncertainties introduced by diverse events as fires, pests, climate changes, government policies, stock prices etc. (Brannstrom & Sumpter, 2006). Very often these events might have long-range or short-range consequences on biological system. To take into account both types of consequences and to describe renewable resource stock dynamics it is reasonable to use stochastic differential equation (SDE) with fractional Brownian motion (fBm):

$$dX(t) = f\left(t, X(t), u(t)\right)dt + \sum_{i=1}^{n} q_i\left(t, X(t)\right)dB_t^{\mathsf{H}_i}, \quad X(t_0) = X_0, \tag{5}$$

where $f\left(t, X(t), u(t)\right) := s\left(t, X(t)\right) - u(t)$ and $q_i\left(t, X(t)\right)$ are smooth functions, $dB_t^{\mathsf{H}_i}$ are uncorrelated increments of fBm with the Hurst parameters $\mathsf{H}_i \in (0,1)$ in the sense that

$$X(t) = X_0 + \int_{t_0}^{t} f(\tau, X(\tau), u(\tau)) d\tau + \sum_{i=1}^{k} \int_{t_0}^{t} q_i(\tau, X(\tau), u(\tau)) dB_\tau^{H_i} , \tag{6}$$

where second integral can be understand as a pathwise integral or as a stochastic Skorokhod integral with respect to the fBm.

An economical component of the bioeconomic model can be introduced as discounted value of utility function or production function, which may involve three types of input, namely labor $L(t)$, capital $C(t)$ and natural resources $X(t)$:

$$F(t, X(t), u(t)) = e^{-\rho t} \Pi\left(L^{\gamma_L}(t), C^{\gamma_C}(t), X^{\gamma}(t)\right), \tag{7}$$

where $\Pi\left(L^{\gamma_L}(t), C^{\gamma_C}(t), X^{\gamma}(t)\right)$ is the multiplicative Cobb-Douglas function with γ_L, γ_C and γ constant of elasticity, which corresponds to the net revenue function at time t from having a resource stock of size $X(t)$ and harvest $u(t)$, ρ is the annual discount rate.

The model (7) was used in (Filatova & Grzywaczewski, 2009) for named task solution, other production function models can be found, for an example in (Kugarajh et al., 2006) or (Gonzalez-Olivares, 2005)):

$$F(t, X(t), u(t)) = e^{-\rho t} \Pi(C(t), X(t)) = e^{-\rho t}\left[p(t, u(t)) - c(t, X(t), u(t))\right], \tag{8}$$

where $p(\cdot,\cdot)$ is the inverse demand function and $c(\cdot,\cdot,\cdot)$ is the cost function.

In both cases the objective of the management is to maximize the expected utility

$$J(X(\cdot), u(\cdot)) = \max_{u(t)} \mathsf{E}\left[\int_{t_0}^{t_1} F(t, X(t), u(t)) dt\right] \tag{9}$$

on time interval $[t_0, t_1]$ **subject to** constraints (4) and (5), where $\mathsf{E}[\cdot]$ is mathematical expectation operator.

The problem (4), (5), (9) could be solved by means of maximum principle staying with the idea of MSY. There are several approaches, which allow find optimal harvest rate. First group operates in terms of stochastic control (Yong, 1999) and (Biagini et al., 2002), second one is based on converting the task (9) to non-random fractional optimal control (Jumarie, 2003). It is also possible to use system of moments equations instead of equation (5) as it was proposed in (Krishnarajaha et al., 2005) and (Lloyd, 2004). Unfortunately, there are some limitations, namely the redefinition of MSY for the model (5) and in a consequence finding an optimal harvest cannot be done by classical approaches (Bousquet et al., 2008) and numerical solution for stochastic control problems is highly complicated even for linear SDEs.

To overcome these obstacles we propose to combine the production functions (7) and (8) using $\mathsf{E}\left[X^{\gamma}(t)\right]$ instead of $\mathsf{E}\left[X(t)\right]$ in the function (8), specifically the goal function (9) takes a form

$$J(X(\cdot),u(\cdot)) = \max_{u(t)} \int_{t_0}^{t_1} F\left(t, \mathsf{E}\left[X^{\gamma}(t)\right], u(t)\right) dt \,, \tag{10}$$

where $\gamma \in (0,1]$.

If the coefficient of elasticity $\gamma = 1$, then the transformation to a non-random task gives a possibility to apply the classical maximum principle. If $0 < \gamma < 1$, then the cost function (8) contains a fractional term, which requires some additional transformations. This allows to introduce an analogue of MSY taking into account multiplicative environmental noises, as it was mentioned in *Introduction*, in the following manner

$$X^{*} = \max \mathsf{E}\left[X^{\gamma}(t)\right], \tag{11}$$

which can be treated as the state constraint.

Now the optimal harvest task can be summarized as follows. The goal is to maximize the utility function (10) subject to constraints (4), (5), and (11).

2.2 A background of dynamic fractional moment equations

To get an analytical expression for $\mathsf{E}\left[X^{\gamma}(t)\right]$ it is required to complete some transformations. The fractal terms complicate the classical way of the task solution and therefore some appropriate expansion of fractional order is required even if it gives an approximation of dynamic fractional moment equation. In the next reasoning we will use ideas of the fractional difference filters. The basic properties of the fractional Brownian motion can be summarized as follows (Shiryaev, 1998).

Definition. Let $(\Omega, \mathsf{F}, \mathsf{P})$ denotes a probability space and H, $0 < \mathsf{H} < 1$, referred to as the Hurst parameter. A centered Gaussian process $B^{\mathsf{H}} = \left\{B(t,\mathsf{H}), t \geq 0\right\}$ defined on this probability space is a fractional Brownian motion of order H if

$$\mathsf{P}\left\{B(0,\mathsf{H}) = 0\right\} = 1$$

and for any $t, \tau \in \mathsf{R}^{+}$

$$\mathsf{E}\left\{B(t,\mathsf{H})B(\tau,\mathsf{H})\right\} = \tfrac{1}{2}\left(t^{2\mathsf{H}} + \tau^{2\mathsf{H}} - \left|t-\tau\right|^{2\mathsf{H}}\right).$$

If $\mathsf{H} = \frac{1}{2}$, B^{H} is the ordinary Brownian motion.

There are several models of fractional Brownian motion. We will use Maruyama's notation for the model introduced in (Mandelbrot & Van Ness, 1968) in terms of Liouville fractional derivative of order H of Gaussian white noise. In this case, the fBm increment of (5) can be written as

$$dB_t^{\mathsf{H}} = \omega(t)(dt)^{\mathsf{H}} \tag{12}$$

where $\omega(t)$ is the Gaussian random variable.

Now the equation (5) takes a form

$$dX(t) = f\left(t, X(t), u(t)\right)dt + \sum_{p=1}^{n} q_p\left(t, X(t)\right)\omega_p(t)(dt)^{\mathsf{H}_p} . \tag{13}$$

The results received in (Jumarie, 2007) allow to obtain the dynamical moments equations

$$m_k := \mathsf{E}\left\{X^k(t)\right\} \equiv \left\langle X^k(t)\right\rangle , \tag{14}$$

where $k \in N^*$.

Using the equality

$$X(t+dt) = X(t) + dX , \tag{15}$$

we get the following relation

$$X^k(t+dt) = X^k(t) + \sum_{j=1}^{k} \binom{k}{j} X^{k-j}(t)(dX)^j , \tag{16}$$

with

$$(dX)^j = \left(f\left(t, X(t), u(t)\right)dt + \sum_{i=1}^{n} q_i\left(t, X(t)\right)dB_i \right)^j ,$$

where $dB_i := dB^{\mathsf{H}_i}$.

Taking the mathematical expectation of (16) yields the equality

$$m_k(t+dt) = m_k(t) + \sum_{j=1}^{k} \binom{k}{j} \mathsf{E}\left\{X^{k-j}(t)\left(dX(t)\right)^j\right\} . \tag{17}$$

In order to obtain the explicit expression of (17) we suppose that random variables ω_i and ω_j are uncorrelated for any $i \neq j$ and denote $\eta(v) = \frac{1}{2n} v^{2\ell} = \frac{1}{2n}\left(\omega(t)(dt)^{\mathrm{H}}\right)^{2\ell}$ for arbitrary integer ℓ. Application of the Ito formula gives

$$\frac{1}{2n} v_t^{2\ell} - \frac{1}{2n} v_0^{2\ell} = \int_0^t v_s^{2\ell-1} dv_s + \frac{1}{2}\int_0^t (2n-1) v_s^{2\ell-2} dv_s \,. \tag{18}$$

Taking expectation and solving (18) in iterative manner, we get the following results

$$\begin{aligned}
\mathsf{E}\left(v_t^{2\ell}\right) &= \tfrac{2\ell(2\ell-1)}{2}\int_0^t \mathsf{E}\left(v_s^{2\ell-2}\right) ds \\
&= \tfrac{2\ell(2\ell-1)}{2}\tfrac{(2\ell-2)(2\ell-3)}{2}\int_0^t\int_0^{t_1} \mathsf{E}\left(v_s^{2\ell-4}\right) ds\, dt_1 \\
&\vdots \\
&= \tfrac{(2\ell)!}{2^\ell}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{\ell-1}} 1\, ds\, dt_{\ell-1} dt_1
\end{aligned}$$

Successive solution of this expression brings the sequence $t_{\ell-1}$, $\frac{1}{2!}t_{\ell-2}^2$, $\frac{1}{3!}t_{\ell-3}^3$,..., $\frac{1}{\ell!}t_0^\ell$ and gives the expression for even moments

$$\mathsf{E}\left[\left(\omega(t)(dt)^{\mathrm{H}}\right)^{2\ell}\right] = \frac{(2\ell)!}{\ell! 2^\ell}(dt)^{2\ell H} \,.$$

The same can be done to get odd moments, namely

$$\mathsf{E}\left\{\left(\omega(t)(dt)^{\mathrm{H}}\right)^{2\ell+1}\right\} = 0 \,.$$

Now (17) can be presented in the following way:

$$m_k(t+dt) = m_k(t) + k\left\langle X^{k-1} dX\right\rangle + \frac{k(k-1)}{2}\left\langle X^{k-2} dX^2\right\rangle + \mathsf{O}\left(dt^{1+\varepsilon}\right),$$

for $k \in N^*$ and $\varepsilon > 0$.

Let L denote the lag operator and γ be the fractional difference parameter. In this case the fractional difference filter $(1-\mathsf{L})^\gamma$ is defined by a hypergeometric function as follows (Tarasov, 2006)

$$(1-\mathsf{L})^\gamma = \sum_{k=0}^{\infty} \frac{\Gamma(k-\gamma)}{\left(\Gamma(-\gamma)\Gamma(k+1)\right)} \mathsf{L}^\gamma \,, \tag{19}$$

where $\Gamma(\cdot)$ is the Gamma function.

Right hand-side of (19) can be also approximated by binominal expansion

$$\left(1-\mathrm{L}\right)^{\gamma} \approx 1-\gamma\mathrm{L} + \frac{\gamma(\gamma-1)}{2!}\mathrm{L}^{2} - \frac{\gamma(\gamma-1)(\gamma-2)}{3!}\mathrm{L}^{3} + \ldots$$

This expansion allows to rewrite (17) and finally to get an approximation of dynamic fractional moment equation of order γ

$$dm_{\gamma}(t) = \gamma f\left(t, m_{\gamma}(t), u(t)\right)dt + \frac{\gamma(\gamma-1)}{2}\left(q\left(t, m_{\gamma}^{1/2}(t)\right)\right)^{2}(dt)^{2\mathrm{H}}, \tag{20}$$

where $m_{\gamma}(t_0) = \mathrm{E}\left[X^{\gamma}(t_0)\right]$.

To illustrate the dynamic fractional moment equation (20) we will use the following SDE

$$dX(t) = \theta_1 X(t)\left(1-\theta_2 X(t)\right)dt + \theta_3 X(t)\, dB_t^{\mathrm{H}}, \tag{21}$$

where $X(t_0) = 25000$, $\theta_1 = 0.2246$, $\theta_2 = \frac{1}{564795}$, $\theta_3 = 0.0002$ and $\mathrm{H} = 0.5$.

Applying (20) to (21) and using a set of $\gamma \in \{0.25; 0.5; 0.75; 0.95; 1\}$, we can see possible changes in population size (Fig.3) and select the appropriate risk aversion coefficient γ.

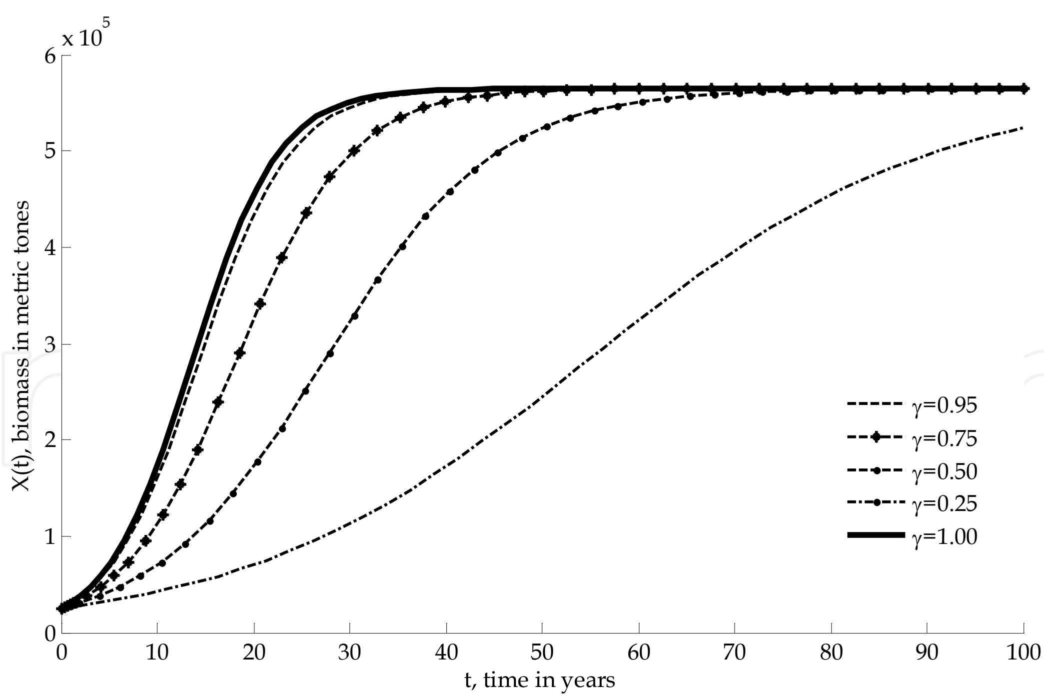

Fig. 3. The dynamic fractional moment equation (20) for equation (21)

2.3 Some required transformations

To get rid of fractional term $(dt)^{2\mathrm{H}}$ and to obtain more convenient formulations of the results we replace ordinary fractional differential equation (20) by integral one

$$x(t)-x(t_0)=\int_{t_0}^{t} f(\tau,x(\tau),u(\tau))d\tau + \int_{t_0}^{t} q(\tau,x(\tau))(d\tau)^{2\mathrm{H}}, \tag{22}$$

where $x(t):=m_\gamma(t)$, $x(t_0):=m_\gamma(t_0)$ for arbitrary selected γ.

Following reasoning is strongly dependent on H value as far as it changes the role of integration with respect to fractional term, namely as in (Jumarie, 2007), denoting the kernel by $\kappa(\tau)$, one has for $0<\mathrm{H}<\frac{1}{2}$

$$\int_{t_0}^{t} \kappa(\tau)(d\tau)^{2\mathrm{H}} = 2\mathrm{H}\int_{t_0}^{t} (t-\tau)^{2\mathrm{H}-1}\kappa(\tau)d\tau, \tag{23}$$

and for $\frac{1}{2}<\mathrm{H}<1$

$$\int_{t_0}^{t} \kappa(\tau)(d\tau)^{2\mathrm{H}} = \mathrm{H}^2\left[\int_{t_0}^{t} (t-\tau)^{\mathrm{H}-1}\kappa^{1/2}(\tau)d\tau\right]^2. \tag{24}$$

So, if $0<\mathrm{H}<\frac{1}{2}$, then the equation (22) can be rewritten as

$$x(t)-x(t_0)=\int_{t_0}^{t} f(\tau,x(\tau),u(\tau))d\tau, + 2\mathrm{H}\int_{t_0}^{t} \frac{1}{(t-\tau)^{1-2\mathrm{H}}} q(\tau,x(\tau))d\tau \tag{25}$$

for $\frac{1}{2}<\mathrm{H}<1$ equation (22) takes the form

$$x(t)-x(t_0)=\int_{t_0}^{t} f(\tau,x(\tau),u(\tau))d\tau + \left[\mathrm{H}\int_{t_0}^{t} \frac{\sqrt{q(\tau,x(\tau))}}{(t-\tau)^{1-\mathrm{H}}}d\tau\right]^2. \tag{26}$$

3. Local maximum principle

3.1 Statement of the problem

Let the time interval $[t_0, t_1]$ be fixed, $x \in \mathrm{R}$ denote the state variable, and $u \in \mathrm{R}$ denote the control variable. The coast function has the form

$$J(x(\cdot), u(\cdot)) = \left\{ \int_{t_0}^{t_1} F((t, x(t), u(t))\, dt + \varphi(x(t_1)) \right\} \to \max_{u(t)}, \tag{27}$$

where F and φ are smooth (C^1) functions, and is subjected to the constraints:

- the object equation (equality constraint)

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f\left(\tau, x(\tau), u(\tau)\right) d\tau \\ &+ \gamma(1-\gamma)\mathrm{H}_1 \int_{t_0}^{t} \frac{q_1\left(\tau, x(\tau)\right)}{(t-\tau)^{1-2\mathrm{H}_1}} d\tau + \left[\mathrm{H}_2 \int_{t_0}^{t} \frac{\sqrt{\frac{1}{2}\gamma(1-\gamma) q_2\left(\tau, x(\tau)\right)}}{(t-\tau)^{1-\mathrm{H}_2}} d\tau \right]^2, \end{aligned} \tag{28}$$

where initial condition $x(t_0) = a > 0$ ($a \in \mathrm{R}$), $\mathrm{H}_1 \in (0, 0.5]$ and $\mathrm{H}_2 \in (0.5, 1.0)$,

- the control constraint (inequality constraint)

$$\phi(u(t)) \le 0, \tag{29}$$

where $\phi(u)$ is a smooth (C^1) vector function of the dimension $\mathbf{p}$,

- the state constraint (inequality constraint)

$$\Phi(x(t)) \le 0, \tag{30}$$

where $\Phi(x)$ is a smooth (C^1) function of the dimension $\mathbf{q}$.

Consider a more general system of integral equations than (28) with condition (30) (particularly $x(t) \ge 0$)

$$x(t) = \int_{t_0}^{t} f(\tau, x(\tau), u(\tau))\, d\tau + \gamma(1-\gamma)\mathrm{H}_1 \int_{t_0}^{t} \frac{\theta_3 x(\tau)}{(t-\tau)^{1-2\mathrm{H}_1}} d\tau + G(y(t)), \tag{31}$$

$$y(t) = b + \int_{t_0}^{t} \frac{g(x(\tau))}{(t-\tau)^{1-\mathrm{H}_2}} d\tau, \tag{32}$$

where $x \in \mathrm{R}^n$, $y \in \mathrm{R}^m$, $u \in \mathrm{R}^r$, $b \in \mathrm{R}^m$, $g(x)$ and $G(x)$ are smooth (C^1) functions.

In addition,

$$(t,x,y,u) \in \mathbf{Q}, \tag{33}$$

where $\mathbf{Q}$ is an open set.
So, we study problem (27), (29) - (33).

3.2. Derivation of the local maximum principle

Set $k := \gamma(1-\gamma)\mathrm{H}_1\theta_3$, $\gamma_1 := 1-2\mathrm{H}_1$, and $\gamma_2 := 1-\mathrm{H}_2$. Define a nonlinear operator

$$P : (x,y,u) \in C \times C \times L^\infty \to (z,\zeta) \in C \times C,$$

where

$$z(t) = x(t) - \int_{t_0}^{t} f(\tau, x(\tau), u(\tau))\,d\tau - k\int_{t_0}^{t} \frac{x(\tau)}{(t-\tau)^{\gamma_1}}\,d\tau - G(y(t))$$

and

$$\zeta(t) = y(t) - b - \int_{t_0}^{t} \frac{g(x(\tau))}{(t-\tau)^{\gamma_2}}\,d\tau.$$

The equation $P(x,y,u)=0$ is equivalent to the system (31) - (32). Let (x,y,u) be an admissible point in the problem. We assume that $\Phi(x(t_0)) < 0$ and $\Phi(x((t_1)) < 0$. The derivative of P at the point (x,y,u) is a linear operator

$$P'(x,y,u) : (\bar{x},\bar{y},\bar{u}) \to (\bar{z},\bar{\zeta}),$$

where

$$\begin{aligned}
\bar{z}(t) &= \bar{x}(t) - \int_{t_0}^{t} f_x(\tau, x(\tau), u(\tau))\bar{x}(\tau)\,d\tau \\
&\quad - \int_{t_0}^{t} f_u(\tau, x(\tau), u(\tau))\bar{u}(\tau)\,d\tau \\
&\quad - k\int_{t_0}^{t} \frac{\bar{x}(\tau)}{(t-\tau)^{\gamma_1}}\,d\tau - G'(y(t))\bar{y}(t), \\
\bar{\zeta}(t) &= \bar{y}(t) - \int_{t_0}^{t} \frac{g'(x(\tau))\bar{x}(\tau)}{(t-\tau)^{\gamma_2}}\,d\tau.
\end{aligned}$$

Set $f_x(\tau) := f_x(\tau, x(\tau), u(\tau))$ $f_u(\tau) := f_u(\tau, x(\tau), u(\tau))$ etc. An arbitrary linear functional ℓ, vanishing on the kernel of the operator $P'(x, y, u)$, has the form

$$\ell(\overline{x},\overline{y},\overline{u}) = \int_{t_0}^{t_1} \overline{x}(t)\, d\sigma_1(t) -$$
$$-\int_{t_0}^{t_1} \left[\int_{t_0}^{t} \left(f_x(\tau)\overline{x}(\tau) + f_u(\tau)\overline{u}(\tau) \right) d\tau \right] d\sigma_1(t) -$$
$$-k\int_{t_0}^{t_1} \left[\int_{t_0}^{t} \frac{\overline{x}(\tau)}{(t-\tau)^{\gamma_1}} d\tau \right] d\sigma_1(t) - \int_{t_0}^{t_1} G'\left(y(t)\right)\overline{y}(t)\, d\sigma_1(t) +$$
$$+\int_{t_0}^{t_1} \overline{y}(t)\, d\sigma_2(t) - \int_{t_0}^{t_1} \left[\int_{t_0}^{t} \frac{g'(\tau)\overline{x}(\tau)}{(t-\tau)^{\gamma_2}} d\tau \right] d\sigma_2(t).$$

We change the order of integrating

$$\ell(\overline{x},\overline{y},\overline{u}) = \int_{t_0}^{t_1} \overline{x}(t)\, d\sigma_1(t) -$$
$$-\int_{t_0}^{t_1} \left[\int_{\tau}^{t_1} \left(f_x(\tau)\overline{x}(\tau) + f_u(\tau)\overline{u}(\tau) \right) d\sigma_1(t) \right] d\tau -$$
$$-k\int_{t_0}^{t_1} \left[\int_{\tau}^{t_1} \frac{\overline{x}(\tau)}{(t-\tau)^{\gamma_1}} d\sigma_1(t) \right] d\tau - \int_{t_0}^{t_1} G'\left(y(t)\right)\overline{y}(t)\, d\sigma_1(t) +$$
$$+\int_{t_0}^{t_1} \overline{y}(t)\, d\sigma_2(t) - \int_{t_0}^{t_1} \left[\int_{\tau}^{t_1} \frac{g'(\tau)\overline{x}(\tau)}{(t-\tau)^{\gamma_2}} d\sigma_2(t) \right] d\tau .$$

We now replace τ by t and t by τ and get

$$\ell(\overline{x},\overline{y},\overline{u}) = \int_{t_0}^{t_1} \overline{x}(t)\, d\sigma_1(t) -$$
$$-\int_{t_0}^{t_1} \left[\int_{t}^{t_1} \left(f_x(t)\overline{x}(t) + f_u(t)\overline{u}(t) \right) d\sigma_1(\tau) \right] dt -$$
$$-k\int_{t_0}^{t_1} \left[\int_{t}^{t_1} \frac{\overline{x}(t)}{(\tau-t)^{\gamma_1}} d\sigma_1(\tau) \right] dt - \int_{t_0}^{t_1} G'\left(y(t)\right)\overline{y}(t)\, d\sigma_1(t) +$$
$$+\int_{t_0}^{t_1} \overline{y}(t)\, d\sigma_2(t) - \int_{t_0}^{t_1} \left[\int_{t}^{t_1} \frac{g'(t)\overline{x}(t)}{(\tau-t)^{\gamma_2}} d\sigma_2(\tau) \right] dt$$

The Euler equation has the form

$$
\begin{gathered}
-\alpha_0 \int_{t_0}^{t_1} \left(F_x(t)\overline{x}(t) + F_u(t)\overline{u}(t) \right) dt \\
-\alpha_0 \varphi'\left(x(t_1)\right)\overline{x}(t_1) \\
+\ell(\overline{x},\overline{y},\overline{u}) + \left\langle \lambda, \phi'\left(u(\cdot)\right)\overline{u}(\cdot) \right\rangle \\
+\int_{t_0}^{t_1} \Phi'\left(x(t)\right)\overline{x}(t)\, d\mu(t) = 0,
\end{gathered}
$$

where $\lambda \in (L^\infty)^*$, $\lambda \geq 0$, $\left\langle \lambda, \phi\left(u(\cdot)\right) \right\rangle = 0$, $d\mu \in C^*$, $d\mu \geq 0$, $\Phi\left(x(t)\right) d\mu(t) = 0$.

Note that the complementary slackness condition $\Phi\left(x(t)\right) d\mu(t) = 0$ combined with the assumptions $\Phi\left(x(t_0)\right) < 0$ and $\Phi\left(x(t_1)\right) < 0$ imply that the measure $d\mu$ is zero in some neighborhoods of the points t_0 and t_1.

Setting in the Euler equation $\overline{x} = 0$, $\overline{y} = 0$, we get

$$
-\alpha_0 F_u(t) - f_u(t) \int_t^{t_1} d\sigma_1(\tau) + \lambda^a(t)\phi'\left(u(t)\right) = 0, \tag{34}
$$

where λ^a is an absolutely continuous part of λ. Hence $\lambda^a(\cdot) \geq 0$, $\lambda^a(\cdot)\phi\left(u(\cdot)\right) = 0$.
Setting $\overline{u} = 0$, $\overline{y} = 0$, we get

$$
\begin{gathered}
-\alpha_0 F_x(t)\,dt - \alpha_0 \varphi'\left(x(t_1)\right)\delta(t - t_1)\,dt + d\sigma_1(t) \\
-f_x(t)\left[\int_t^{t_1} d\sigma_1(\tau)\right] dt - k\left[\int_t^{t_1} \frac{d\sigma_1(\tau)}{(\tau - t)^{\gamma_1}}\right] dt - \\
-g'\left(x(t)\right)\left[\int_t^{t_1} \frac{d\sigma_2(\tau)}{(\tau - t)^{\gamma_2}}\right] dt \quad . \\
+\Phi'\left(x(t)\right) d\mu(t) = 0
\end{gathered} \tag{35}
$$

Finally, setting $\overline{u} = 0$, $\overline{x} = 0$, we get

$$
-G'\left(y(t)\right) d\sigma_1(t) + d\sigma_2(t) = 0 .
$$

From equation (35) we get

$$d\sigma_1(t) = s_1(t)dt + \alpha_0\varphi'(x(t_1))\delta(t-t_1)dt - \Phi'(x(t))d\mu(t)$$

where $s_1 \in L^1$. Set

$$\psi(t) = \int_t^{t_1} d\sigma_1(\tau).$$

Then

$$\psi(t_1) = \alpha_0\varphi'(x(t_1)),$$
$$d\sigma_1(t) = -d\psi(t) + \psi(t_1)\delta(t-t_1)dt,$$
$$d\sigma_2(t) = -G'(y(t))d\psi(t) + \psi(t_1)G'(y(t))\delta(t-t_1)dt.$$

Consequently, we have

$$\int_t^{t_1} \frac{d\sigma_1(\tau)}{(\tau-t)^{\gamma_1}} = -\int_t^{t_1} \frac{d\psi(\tau)}{(\tau-t)^{\gamma_1}} + \psi(t_1)\int_t^{t_1} \frac{\delta(\tau-t_1)d\tau}{(\tau-t)^{\gamma_1}} =$$
$$= -\int_t^{t_1} \frac{d\psi(\tau)}{(\tau-t)^{\gamma_1}} + \frac{\psi(t_1)}{(t_1-t)^{\gamma_1}},$$
$$\int_t^{t_1} \frac{d\sigma_2(\tau)}{(\tau-t)^{\gamma_2}} = -\int_t^{t_1} \frac{G'(y(\tau))d\psi(\tau)}{(\tau-t)^{\gamma_2}} + \psi(t_1)\int_t^{t_1} \frac{G'(y(\tau))\delta(\tau-t_1)d\tau}{(\tau-t)^{\gamma_2}} =$$
$$= -\int_t^{t_1} \frac{G'(y(\tau))d\psi(\tau)}{(\tau-t)^{\gamma_2}} + \psi(t_1)\frac{G'(y(t_1))}{(t_1-t)^{\gamma_2}}.$$

Therefore, relation (34) implies the following local maximum principle:

$$-\alpha_0 F_u(t) - f_u(t)\psi(t) + \lambda^a(t)\phi'(u(t)) = 0 \tag{36}$$

and equation (35) leads to the following adjoint equation

$$\begin{gathered}
-\alpha_0 F_x(t)dt - d\psi(t) - f_x(t)\psi(t)dt \\
+k\left[\int_t^{t_1} \frac{d\psi(\tau)}{(\tau-t)^{\gamma_1}} - \frac{\psi(t_1)}{(t_1-t)^{\gamma_1}}\right]dt \\
+g'(x(t))\int_t^{t_1} \frac{G'(y(\tau))}{(\tau-t)^{\gamma_2}}d\psi(\tau) - g'(x(t))\psi(t_1)\frac{G'(y(t_1))}{(t_1-t)^{\gamma_2}} + \Phi'(x(t))d\mu(t) = 0.
\end{gathered} \tag{37}$$

Thus the following theorem is proved.

Theorem. *Let* $(x(t),y(t),u(t))$ *be {the} [an] optimal process on the interval* $[t_0,t_1]$, *where* $x(\cdot)\in C([t_0,t_1],\mathrm{R}^n)$, $y(\cdot)\in C([t_0,t_1],\mathrm{R}^m)$, $u(\cdot)\in L^\infty([t_0,t_1],\mathrm{R}^r)$. *Then there exists a set of Lagrange multipliers* $(\alpha_0,\psi(\cdot),\lambda(\cdot),\mu)$ *such that* α_0 *is a scalar,* $\psi(\cdot):[t_0,t_1]\to\mathrm{R}^n$ *is a function of bounded variation continuous from the left, defining the measure* $d\psi$, $\lambda(\cdot):[t_0,t_1]\to\mathrm{R}^{r^*}$ *is an integrable function,* $\mu(\cdot):[t_0,t_1]\to\mathrm{R}$ *is a function of bounded variation continuous from the left, defining the measure* $d\mu$ *, and the following conditions are fulfilled:*

(a) nonnegativity: $\alpha_0\ge 0$, $\lambda(t)\ge 0$ *a.e. on* $[t_0,t_1]$, $d\mu\ge 0$;
(b) nontriviality:

$$\alpha_0+\|\psi\|+\|d\mu\|>0\,;$$

(c) complementarity:

$$\lambda(t)\phi(u(t))=0 \quad \text{a.e. on } [t_0,t_1]\,,$$
$$\Phi(x(t))d\mu(t)=0\,;$$

(d) adjoint equation:

$$\begin{aligned}-d\psi(t) \;=\; & \psi(t)f_x(t)dt-k\left[\int_t^{t_1}\frac{d\psi(\tau)}{(\tau-t)^{\gamma_1}}-\frac{\psi(t_1)}{(t_1-t)^{\gamma_1}}\right]dt\\ & -\left[\int_t^{t_1}\frac{G'(y(\tau))d\psi(\tau)}{(\tau-t)^{\gamma_2}}-\frac{\psi(t_1)G'(y(t_1))}{(t_1-t)^{\gamma_2}}\right]g'(x(t))dt+\\ & +\alpha_0F_x(t,x(t),u(t))dt-\Phi'(x(t))d\mu(t)\,,\end{aligned}$$

[where $k:=\gamma(1-\gamma)\mathrm{H}_1\theta_3$, $\gamma_1:=1-2\mathrm{H}_1$, and $\gamma_2:=1-\mathrm{H}_2$;]

(e) transversality condition:

$$\psi(t_1)=\alpha_0\varphi'(x(t_1))\,;$$

(f) local maximum principle:

$$\psi(t)f_u(t,x(t),u(t))+\alpha_0F_u(t,x(t),u(t))-\lambda(t)\phi'(u(t))=0\,.$$

4. Example

In this section we will illustrate the theoretical results to get optimal control for the North-East Arctic Cod Fishery, using partly the data presented in (Kugarajh et al., 2006), by means of the expected utility from terminal wealth maximization and without paying attention on economics and biological aspects of the problem.

Figure 4 shows the biomass time series made by individual vessels. In order to introduce the model for the data description we found the parameters of fBm, using methodology presented in (Filatova, 2008). There was only one significant parameter $\mathsf{H} = 0.4501$ (with standard deviation 0.0073), which allowed to select a model of the biomass population, namely

$$dX_t = \theta_1\left(X_t - \theta_2 X_t^2\right)dt + \theta_3 X_t dB_t^{\mathsf{H}}, \tag{38}$$

where $\mathsf{H} \in \left(0, 0.5\right]$ and $X_0 = X\left(t_0\right)$.

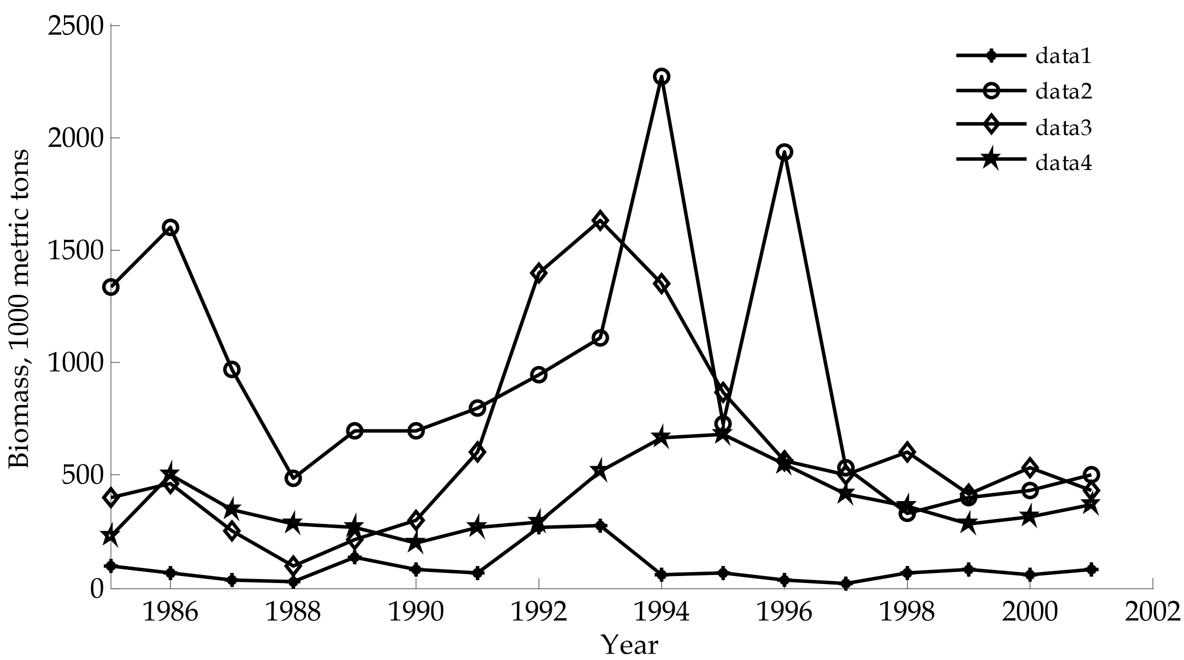

Fig. 4. The North-East Arctic cod biomass for the years 1985 – 2001.

Next to find estimates of (38) we used ideas of identification methods (Filatova & Grzywaczewski, 2007; Filatova et al., 2007) and got

$$dX_t = 0.6416\left(X_t - \tfrac{1}{1567700.1215} X_t^2\right)dt + 0.0031 X_t dB_t^{0.4501}, \tag{39}$$

where initial value $X_0 = 500 \cdot 10^3$.

Applying the goodness-of-fit test for received SDE model (this test can be found in (Allen, 2007)) we calculated for $M = 18$ simulations the test statistics Q =5.0912. Since three parameters were estimated on initial data stock, the number of degree of freedom is $M - 3 = 15$ and the critical value of $\chi^2\left(0.05;15\right) = 24.9958$. The probability $\mathsf{p}\left(\chi^2\left(0.05;15\right) \geq 5.0912\right) < 0.5491$ is greater than the level of significance 0.05. That is, we cannot reject that SDE model (39) describes the biomass dynamics. Thus, we can use the methodology proposed in this work in order to find the optimal strategy.

The model (39) can be used for the forecast of biomass dynamics (see Fig.5). Since the data had a significant variation, it is reasonable to take $\gamma = 0.8$.

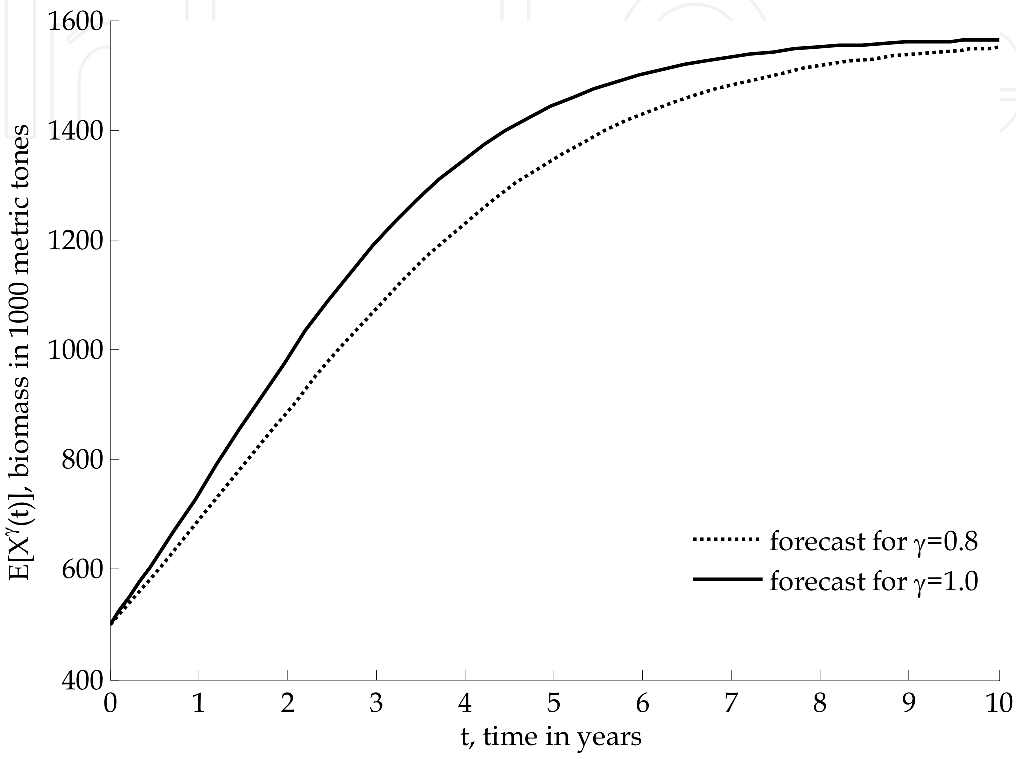

Fig. 5. The 10 years forecast for the North-East Arctic cod biomass for model (39).

Setting $x(t) := \mathsf{E}\left[X^{\gamma}(t)\right]$ and applying transformation (20) we get the object equation (28) in the following form

$$dx_t = \gamma\left[\theta_1\left(x_t - \theta_2 x_t^2\right) - u_t\right]dt + \frac{\gamma(\gamma-1)}{2}\theta_3 x_t dt^{\mathsf{H}}, \tag{40}$$

where $x_0 = \mathsf{E}\left[X^{\gamma}(t_0)\right]$.

Next one can set the constraints (29) and (30) as for an example

$$0 < u(t) < 150 \cdot 10^3, \tag{41}$$

$$0 < x(t) < 251.44 \cdot 10^3. \tag{42}$$

Using the integration role (23) the ordinary fractional differential equation (40) can be rewritten as

$$x(t)-x(t_0)=\int_{t_0}^{t}\gamma\left[\theta_1\left(x(\tau)-\theta_2 x^2(\tau)\right)-u(\tau)\right]d\tau$$
$$+\frac{\gamma(\gamma-1)}{2}\int_{t_0}^{t}\frac{\beta}{(t-\tau)^{1-\beta}}\theta_3 x(\tau)d\tau\ , \tag{43}$$

where $\beta = 2\mathrm{H}$.
Next we define the goal function (27) with the production function (8)

$$J(x(\cdot),u(\cdot))=\left\{\int_{t_0}^{t_1}e^{-\rho t}\left[au(t)-bu^2(t)-\frac{c}{x(t)}\right]dt+e^{-\rho t_1}\varphi(x(t_1))\right\}\to\max_{u(t)}\ , \tag{44}$$

where $a = 88.25$, $b = 0.0009$ and $c = 1.633\cdot 10^{11}$ are the parameters of production function (8).
Using (43) and (44) we obtain the adjoint equation (37)

$$\begin{aligned}-d\psi(t)&=\psi(t)\gamma\theta_1\left(1-2\theta_2 x(t)\right)dt+\beta\frac{\gamma(\gamma-1)}{2}\theta_3\left[\frac{\psi(t_1)}{(t_1-t)^{1-\beta}}-\int_t^{t_1}\frac{\psi'(\tau)}{(\tau-t)^{1-\beta}}d\tau\right]dt\\&\quad+\alpha_0 e^{-\rho t}\frac{c}{x^2(t)}dt-\Phi'(x(t))d\mu(t).\end{aligned} \tag{45}$$

Finally the local maximum principle (36) is

$$\alpha_0 e^{-\rho t}\left[a-2bu(t)\right]-\gamma\psi(t)+\lambda^a(t)\varphi'(u)=0\ . \tag{46}$$

On the basis of (44) the optimal control function can be defined as

$$u(t)=\frac{1}{2b}\left(a-\gamma\psi(t)e^{\rho t}+\lambda^a(t)\varphi'(u)e^{\rho t}\right). \tag{47}$$

Substitution of (47) to (43) gives

$$\begin{aligned}x(t)-x(t_0)&=\int_{t_0}^{t}\gamma\left[\theta_1\left(x(\tau)-\theta_2x^2(\tau)\right)-\frac{1}{2b}\left(a+\gamma\psi(\tau)e^{\rho\tau}+\lambda^a(t)\varphi'(u)e^{\rho t}\right)\right]d\tau\\&\quad+\frac{\gamma(\gamma-1)}{2}\int_{t_0}^{t}\frac{\beta}{(t-\tau)^{1-\beta}}\theta_3x(\tau)d\tau.\end{aligned} \tag{48}$$

Solution of the system (45), (48) allows to define the solution of adjoint equation $\psi(t)$, optimal control $u(t)$ and as result the expected utility from terminal wealth (44). The ideas of numerical algorithm for the system (45), (48) are presented in (Filatova et al., 2010), that gives following optimal control (see Fig. 6).

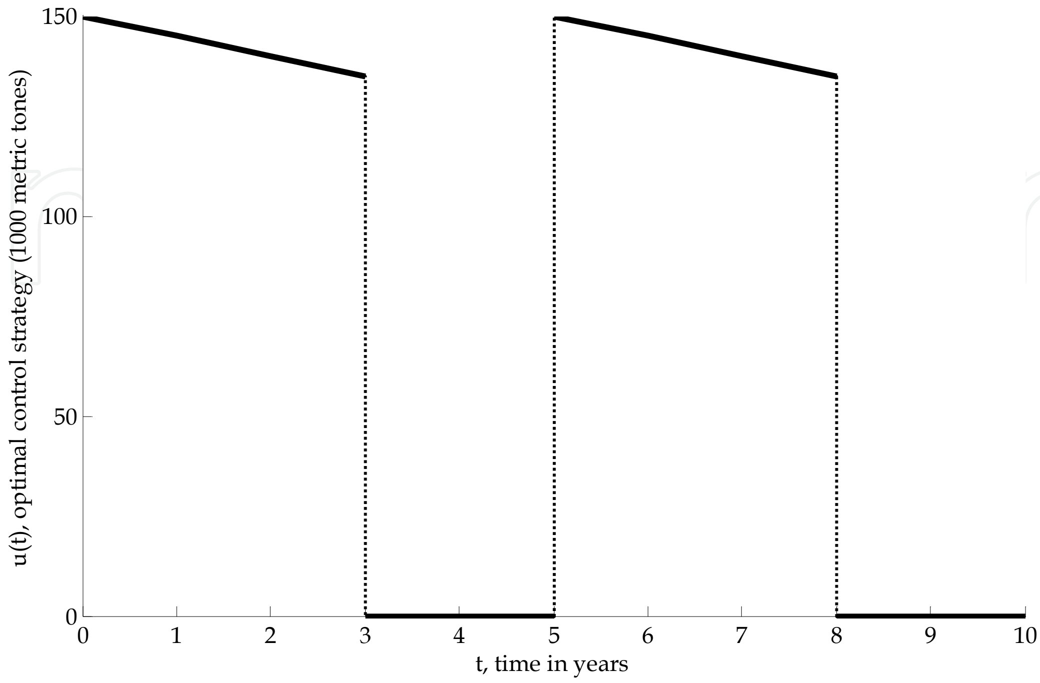

Fig. 6. The optimal control strategy for ten years period for the North-East Arctic cod.

5. Conclusion

In this work we studied stochastic harvest problem, where the biomass dynamics was described by stochastic logarithmic growth model with fractional Brownian motion. Since the data used for the fishery management are not accurate, to maintain existing of the population we proposed to use the risk aversion coefficient for fish stock and added not only control but also state constraints.

This formulation of optimal harvest problem could not be solved by classical methods and required some additional transformations. We used fractional filtration and got the integral object equation, which did not contain stochastic term. As a result stochastic optimization problem was changed to non-random one. Using maximum principle we got necessary optimality conditions, which were used for numerical solution of the North-East Arctic cod fishery problem to set suitable harvest levels.

We hope that to improve the quality of proposed methodology time-varying parameters model can be used as a control object. This requires new parametric identification method from one side and better understanding of economics and biological development of the exploitable ecosystem from the other one.

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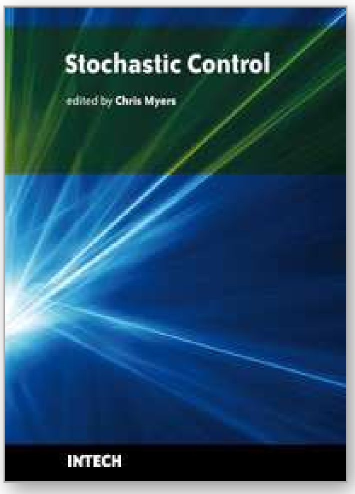

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Uncertainty presents significant challenges in the reasoning about and controlling of complex dynamical systems. To address this challenge, numerous researchers are developing improved methods for stochastic analysis. This book presents a diverse collection of some of the latest research in this important area. In particular, this book gives an overview of some of the theoretical methods and tools for stochastic analysis, and it presents the applications of these methods to problems in systems theory, science, and economics.