5

Zero-sum stopping game associated with threshold probability

Yoshio Ohtsubo
Kochi University
Japan

Abstract

We consider a zero-sum stopping game (Dynkin's game) with a threshold probability criterion in discrete time stochastic processes. We first obtain fundamental characterization of value function of the game and optimal stopping times for both players as the result of the classical Dynkin's game, but the value function of the game and the optimal stopping time for each player depend upon a threshold value. We also give properties of the value function of the game with respect to threshold value. These are applied to an independent model and we explicitly find a value function of the game and optimal stopping times for both players in a special example.

1. Introduction

In the classical Dynkin's game, a standard criterion function is the expected reward (e.g. DynkinDynkin (1969) and NeveuNeveu (1975)). It is, however, known that the criterion is quite insufficient to characterize the decision problem from the point of view of the decision maker and it is necessary to select other criteria to reflect the variability of risk features for the problem (e.g. WhiteWhite (1988)). In a optimal stopping problem, Denardo and RothblumDenardo & Rothblum (1979) consider an optimal stopping problem with an exponential utility function as a criterion function in finite Markov decision chain and use a linear programming to compute an optimal policy. In Kadota et al.Kadota et al. (1996), they investigate an optimal stopping problem with a general utility function in a denumerable Markov chain. They give a sufficient condition for an one-step look ahead (OLA) stopping time to be optimal and characterize a property of an OLA stopping time for risk-averse and risk-seeking utilities. BojdeckiBojdecki (1979) formulates an optimal stopping problem which is concerned with maximizing the probability of a certain event and give necessary and sufficient conditions for existence of an optimal stopping time. He also applies the results to a version of the discrete-time disorder problem. OhtsuboOhtsubo (2003) considers optimal stopping problems with a threshold probability criterion in a Markov process, characterizes optimal values and finds optimal stopping times for finite and infinite horizon cases, and he in Ohtsubo (2003) also investigates optimal stopping problem with analogous objective for discrete time stochastic process and these are applied to a secretary problem, a parking problem and job search problems.

On the other hand, many authors propose a variety of criteria and investigate Markov decision processes for their criteria, instead of standard criteria, that is, the expected discounted total reward and the average expected reward per unit (see WhiteWhite (1988) for survey). Especially, WhiteWhite (1993), Wu and LinWu & Lin (1999), Ohtsubo and ToyonagaOhtsubo & Toyonaga (2002) and OhtsuboOhtsubo (2004) consider a problem in which we minimize a threshold probability. Such a problem is called risk minimizing problem and is available for applications to the percentile of the losses or Value-at-Risk (VaR) in finance (e.g. FilarFilar et al. (1995) and UryasevUryasev (2000)).
In this paper we consider Dynkin's game with a threshold probability in a random sequence. In Section 3 we characterize a value function of game and optimal stopping times for both players and show that the value function of game has properties of a distribution function with respect to a threshold value except a right continuity. In Section 4 we investigate an independent model, as applications of our game, and we explicitly find a value function which is right continuous and optimal stopping times for both players.

2. Formulation of problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n)_{n\in N}$ an increasing family of sub-σ-fields of $\mathcal{F}$, where $N = \{0, 1, 2, \cdots\}$ is a discrete time space. Let $X = (X_n)_{n\in N}, Y = (Y_n)_{n\in N}, W = (W_n)_{n\in N}$ be sequences of random variables defined on $(\Omega, \mathcal{F}, P)$ and adapted to $(\mathcal{F}_n)$ such that $X_n \le W_n \le Y_n$ almost surely (a.s.) for all $n \in N$ and $P(\sup_n X_n^+ + \sup_n Y_n^- < \infty) = 1$, where $x^+ = \max(0, x)$ and $x^- = (-x)^+$. The second assumption holds if random variables $\sup_n X_n^+$ and $\sup_n Y_n^-$ are integrable, which are standard conditions given in the classical Dynkin's game. Also let Z be an arbitrary integrable random variable on $(\Omega, \mathcal{F}, P)$. For each $n \in N$, we denote by Γ_n the class of $(\mathcal{F}_n)$–stopping times τ such that $\tau \ge n$ a. s..
We consider the following zero-sum stopping game. There are two players and the first and the second players choose stopping times τ and σ in Γ_0, respectively. Then the reward paid to the first player from the second is equal to

$$g(\tau, \sigma) = X_\tau I_{(\tau<\sigma)} + Y_\sigma I_{(\sigma<\tau)} + W_\tau I_{(\tau=\sigma<\infty)} + Z I_{(\tau=\sigma=\infty)},$$

where I_A is the indicator function of a set A in $\mathcal{F}$. In the classical Dynkin's game the aim of the first player is to maximize the expected gain $E[g(\tau, \sigma)]$ with respect to $\tau \in \Gamma_0$ and that of the second is to minimize this expectation with respect to $\sigma \in \Gamma_0$. In our problem the objective of the first player is to minimize the threshold probability $P[g(\tau, \sigma) \le r]$ with respect to $\tau \in \Gamma_0$ and the second maximizes the probability with respect to $\sigma \in \Gamma_0$ for a given threshold value r.
We can define processes of minimax and maxmin values corresponding to our problem by

$$\begin{aligned} \overline{V}_n(r) = &\quad \operatorname*{ess\,inf}_{\tau\in\Gamma_n} \operatorname*{ess\,sup}_{\sigma\in\Gamma_n} \; P[g(\tau, \sigma) \le r|\mathcal{F}_n], \\ \underline{V}_n(r) = &\quad \operatorname*{ess\,sup}_{\sigma\in\Gamma_n} \operatorname*{ess\,inf}_{\tau\in\Gamma_n} \; P[g(\tau, \sigma) \le r|\mathcal{F}_n], \end{aligned}$$

respectively, where $P[g(\tau, \sigma) \le r|\mathcal{F}_n]$ is a conditional probability of an event $\{g(\tau, \sigma) \le r\}$ given $\mathcal{F}_n$. See NeveuNeveu (1975) for the definition of ess sup and ess inf. We also define sequences of minimax and maxmin values by

$$\overline{v}_n(r) = \inf_{\tau\in\Gamma_n} \sup_{\sigma\in\Gamma_n} P[g(\tau, \sigma) \le r], \quad \underline{v}_n(r) = \sup_{\sigma\in\Gamma_n} \inf_{\tau\in\Gamma_n} P[g(\tau, \sigma) \le r],$$

respectively. For $n \geq 1$ and $\varepsilon \geq 0$, we say that a pair of stopping times $(\tau_\varepsilon, \sigma_\varepsilon)$ in $\Gamma_n \times \Gamma_n$ is ε–saddle point at (n, r) if

$$P[g(\tau_\varepsilon, \sigma) \leq r] - \varepsilon \leq v_n(r) \leq P[g(\tau, \sigma_\varepsilon) \leq r] + \varepsilon$$

for any $\tau \in \Gamma_n$ and any $\sigma \in \Gamma_n$, when $\overline{v}_n(r) = \underline{v}_n(r)$, say $v_n(r)$. .

3. General results

In this section we give fundamental properties of the value function of the game and find a saddle point.
We notice that $P[g(\tau, \sigma) \leq r] = E[I_{(g(\tau,\sigma) \leq r)}]$ and we easily see that

$$I_{(g(\tau,\sigma)\leq r)} = \widetilde{X}_\tau(r) I_{(\tau<\sigma)} + \widetilde{Y}_\sigma(r) I_{(\sigma<\tau)} + \widetilde{W}_\tau(r) I_{(\tau=\sigma<\infty)} + \widetilde{Z}(r) I_{(\tau=\sigma=\infty)},$$

where new sequences $(\widetilde{X}_n(r)), (\widetilde{Y}_n(r)), (\widetilde{W}_n(r))$ and random variable $\widetilde{Z}(r))$ are defined by

$$\widetilde{X}_n(r) = I_{(X_n \leq r)}, \ \widetilde{Y}_n(r) = I_{(Y_n \leq r)}, \ \widetilde{W}_n(r) = I_{(W_n \leq r)}, \ \widetilde{Z}(r) = I_{(Z \leq r)}.$$

Since $X_n \leq W_n \leq Y_n$, we see that $\widetilde{Y}_n(r) \leq \widetilde{W}_n(r) \leq \widetilde{X}_n(r)$ for all r. Thus our problem is just a special version of the classical Dynkin's game for a *fixed* threshold value r.
We first have three propositions below for a fixed r from the result of Dynkin's game (e.g. see NeveuNeveu (1975) and OhtsuboOhtsubo (2000)). In the following proposition, the notation $\mathrm{mid}(a, b, c)$ denotes the middle value among constants a, b and c. For example, when $a < b < c$ then $\mathrm{mid}(a, b, c) = b$. If $a < b$, $\mathrm{mid}(a, b, c) = \max(a, \min(b, c)) = \min(b, \max(a, c))$.
Proposition 3.1. *Let r be arbitrary.*
(a) *For each $n \in N$, $\overline{V}_n(r) = \underline{V}_n(r)$, say $V_n(r)$, and $\overline{v}_n(r) = \underline{v}_n(r) = E[V_n(r)]$, say $v_n(r)$.*
(b) $(V_n(r))$ *is the unique sequence of random variables satisfying the equalities*

$$V_n = \mathrm{mid}(\widetilde{X}_n(r), \widetilde{Y}_n(r), E[V_{n+1}|\mathcal{F}_n]), \ \ n \in N$$

and the inequalities

$$\widehat{X}_n(r) \leq V_n \leq \widehat{Y}_n(r), \ \ n \in N,$$

where $(\widehat{X}_n(r))$ is the largest submartingale dominated by $\min(\widetilde{X}_n(r), E[\widetilde{Z}(r)|\mathcal{F}_n])$ *and $(\widehat{Y}_n(r))$ is the smallest supermartingale dominating* $\max(\widetilde{Y}_n(r), E[\widetilde{Z}(r)|\mathcal{F}_n])$, *that is,*

$$\widehat{X}_n(r) = \operatorname*{ess\,inf}_{\tau\in\Gamma_n} P[g(\tau, \infty) \leq r)|\mathcal{F}_n], \ \ \widehat{Y}_n(r) = \operatorname*{ess\,sup}_{\sigma\in\Gamma_n} P[g(\infty, \sigma) \leq r|\mathcal{F}_n].$$

(c) *For $\varepsilon > 0$, let*

$$\tau_n^\varepsilon(r) = \inf\{k \geq n | V_k(r) \geq \widetilde{X}_k(r) - \varepsilon\},$$
$$\sigma_n^\varepsilon(r) = \inf\{k \geq n | V_k(r) \leq \widetilde{Y}_k(r) + \varepsilon\}$$

Then $(\tau_n^\varepsilon(r), \sigma_n^\varepsilon(r))$ is ε-saddle point at (n, r).
For the value process $\widehat{X}_n(r)$ for the first player, we can obtain it as the following: for $k \geq n$, let

$$\gamma_k^k(r) = \min\left(\widetilde{X}_k(r), E[\widetilde{Z}(r)|\mathcal{F}_k]\right),$$
$$\gamma_n^k(r) = \max\left(\widetilde{X}_n(r), E[\gamma_{n+1}^k(r)|\mathcal{F}_n]\right), \ \ n < k.$$

Proposition 3.2. *Let r be arbitrary. For each $k, n : k \geq n$, $\gamma_n^k(r) \geq \gamma_n^{k+1}(r)$ and for each $n \in N$, $\lim_{k\to\infty} \gamma_n^k(r) = \widehat{X}_n(r)$.*
For $k \geq n$, let

$$\beta_k^k(r) = \widehat{X}_k(r),$$
$$\beta_n^k(r) = \mathrm{mid}(\widetilde{X}_n(r), \widetilde{Y}_n(r), E[\beta_{n+1}^k(r)|\mathcal{F}_n]),\ n < k,$$

Proposition 3.3. *Let r be arbitrary. For each $k \geq n$, $\beta_n^k(r) \leq \beta_n^{k+1}$ and for each n, $\lim_{k\to\infty} \beta_n^k(r) = V_n(r)$.*
Theorem 3.1. For each n, $V_n(\cdot)$ has properties of a distribution function on R except for the right continuity.
Proof. We first notice that $\widetilde{Z}(r) = I_{(Z\leq r)}$ is a nondecreasing function in r. From the definition of a conditional expectation and the dominated convergence theorem, $E[\widetilde{Z}(r)|\mathcal{F}_k]$ for each k is also nondecreasing at r. Since $\widetilde{X}_k(r) = I_{(X_k\leq r)}$ is nondecreasing at r for each $k \in \mathbf{N}$, we see that $\gamma_k^k(r) = \min(\widetilde{X}_k(r), E[\widetilde{Z}(r)|\mathcal{F}_k])$ is a nondecreasing function in r. By induction, $\gamma_n^k(r)$ is nondecreasing in r for each $k \geq n$. Since a sequence $\{\gamma_n^k(r)\}_{k=n}^{\infty}$ of functions is nonincreasing and $\widehat{X}_n(r) = \lim_{k\to\infty} \gamma_n^k(r)$, it follows that $\beta_n^n(r) = \widehat{X}_n(r)$ is nondecreasing for each n. Similarly, it follows by induction that $\beta_n^k(r)$ is nondecreasing at r for each $n \leq k$, since $\widetilde{Y}_n(r)$ is nondecreasing at r. From Proposition 2.3, the monotonicity of a sequence $\{\beta_n^k(r)\}_{k=n}^{\infty}$ implies that $V_n(r) = \lim_{k\to\infty} \beta_n^k(r)$ is a nondecreasing function in r.
Next, since we have $V_n(r) \leq \widetilde{X}_n(r)$ and we see that $\widetilde{X}_n(r) = I_{(X_n\leq r)} = 0$ for a sufficiently small r, it follows that $\lim_{r\to-\infty} V_n(r) = 0$. Similarly, we see that $\lim_{r\to\infty} V_n(r) = 1$, since we have $V_n(r) \geq \widetilde{Y}_n(r)$ and we see that $\widetilde{Y}_n(r) = 1$ for a sufficiently large r. Thus this theorem is completely proved.
We give an example below in which the value function $V_n(r)$ is not right continuous at some r.
Example 3.1. Let $X_n = W_n = -1, Y_n = 1/n$ for each n and let $Z = 1$. We shall obtain the value function $V_n(r)$ by Propositions 3.2 and 3.3. Since $\widetilde{X}_k(r) = I_{[-1,\infty)}(r)$ and $\widetilde{Z}(r) = I_{[1,\infty)}(r)$, we have $\gamma_k^k(r) = I_{[1,\infty)}(r)$. By induction, we easily see that $\gamma_n^k(r) = I_{[1,\infty)}(r)$ for each $k \geq n$ and hence $\beta_n^n(r) = \widehat{X}_n(r) = \lim_{k\to\infty} \gamma_n^k = I_{[1,\infty)}(r)$. Next, since $\widetilde{Y}_{k-1}(r) = I_{[1/(k-1),\infty)}(r)$, we have $\beta_{k-1}^k(r) = I_{[1/(k-1),\infty)}(r)$. By induction, we see that $\beta_n^k(r) = I_{[1/(k-1),\infty)}(r)$ for each $k > n$. Thus we have $V_n(r) = \lim_{k\to\infty} \beta_n^k(r) = I_{(0,\infty)}(r)$, which yields that $V_n(r)$ is not right continuous at $r = 0$.

4. Independent model

We shall consider an independent sequences as a special model. Let $(W_n)_{n\in N}$ be a sequence of independent distributed random variables with $P(\sup_n |W_n| < \infty) = 1$, and let Z be a random variable which is independent of $(W_n)_{n\in N}$. For each $n \in N$ let $\mathcal{F}_n$ be the σ-field generated by $\{W_k; k \leq n\}$. Also, for each $n \in N$, let $X_n = W_n - c$ and $Y_n = W_n + d$, where c and d are positive constants.
Since $\mathcal{F}_n$ is independent of $\{W_k; k > n\}$, the relation in Proposition 3.1 (b) is represented as follows:

$$\begin{aligned} V_n(r) &= \mathrm{mid}(\widetilde{X}_n(r), \widetilde{Y}_n(r), E[V_{n+1}(r)]) \\ &= \mathrm{mid}(I_{(W_n\leq r+c)}, I_{(W_n\leq r-d)}, E[V_{n+1}(r)]). \end{aligned}$$

From Proposition 3.1 (b) and argument analogous to classical optimal stopping problem, we have also

$$\widehat{X}_n(r) = \min(\widetilde{X}_n(r), E[\widetilde{Z}(r)|\mathcal{F}_n], E[\widehat{X}_{n+1}(r)|\mathcal{F}_n]),$$
$$\widehat{Y}_n(r) = \max(\widetilde{Y}_n(r), E[\widetilde{Z}(r)|\mathcal{F}_n], E[\widehat{Y}_{n+1}(r)|\mathcal{F}_n]).$$

Hence we obtain

$$\widehat{X}_n(r) = \min(\widetilde{X}_n(r), P(Z \le r), E[\widehat{X}_{n+1}(r)]),$$
$$\widehat{Y}_n(r) = \max(\widetilde{Y}_n(r), P(Z \le r), E[\widehat{Y}_{n+1}(r)]),$$

since $E[\widetilde{Z}(r)|\mathcal{F}_n] = E[\widetilde{Z}(r)] = P(Z \le r)$.

Example 4.1. Let W be a uniformly distributed random variable on an interval $[0,1]$ and assume that W_n has the same distribution as W for all $n \in N$ and that $0 < c, d < 1/2$. Then since $(W_n)_{n\in N}$ is a sequence of independently and identically distributed random variables, $V_n(r)$ does not depend on n. Hence, letting $V(r) = V_n(r), n \in N$ and $v(r) = E[V(r)]$, we have

$$V(r) = \mathrm{mid}(I_{(W\le r+c)}, I_{(W\le r-d)}, v(r)).$$

When $W < r - d$, we have $I_{(W\le r+c)} = I_{(W\le r-d)} = 1$, so $V(r) = 1$. When $W \ge r + c$, we have $V(r) = 0$, since $I_{(W\le r+c)} = I_{(W\le r-d)} = 0$. Thus we obtain

$$V(r) = I_{(W\le r-d)} + v(r)I_{(r-d\le W<r+c)}.$$

Taking the expectation on the both sides, we see that

$$v(r) = P(W \le r - d) + v(r)P(r - d \le W < r + c).$$

If $r < d$ then we have $v(r) = v(r)P(0 \le W < r + c)$. Since $r < d < 1/2 < 1 - c$, $P(0 \le W < r + c) < 1$ and hence $v(r) = 0$. If $d \le r < 1 - c$, then we obtain $v(r) = (r - d)/(1 - c - d)$, since $P(W \le r - d) = r - d$ and $P(r - d \le W < r + c) = c + d$. Similarly, if $r \ge 1 - c$ then we have $v(r) = 1$. Thus it follows that

$$v(r) = I_{[1-c,\infty)}(r) + (r - d)/(1 - c - d)I_{[d,1-c)}(r).$$

We completely obtained the values $V(r)$ and $v(r)$. By the way we easily see that $\widehat{X}(r) = \widehat{X}_n(r) = E[\widehat{X}(r)]I_{(W\le r+c)}$, where

$$E[\widehat{X}(r)] = rI_{[1-c,1)}(r) + I_{[1,\infty)}(r),$$

and

$$E[\widehat{Y}(r)] = \widehat{Y}(r) = \widehat{Y}_n(r) = P[Z \le r]I_{(-\infty,d)}(r) + I_{[d,\infty)}(r).$$

Now $v(r)$ is a distribution function in r. Let U is a random variable corresponding to $v(r)$. Then we see that $E[U] = (1 - c + d))/2$.

We shall next compare our model with the classical Dynkin's game in this example. Let

$$\overline{J}_n = \operatorname*{ess\,inf}_{\tau\in\Gamma_n} \operatorname*{ess\,sup}_{\sigma\in\Gamma_n} E[g(\tau,\sigma)|\mathcal{F}_n],$$
$$\underline{J}_n = \operatorname*{ess\,sup}_{\sigma\in\Gamma_n} \operatorname*{ess\,inf}_{\tau\in\Gamma_n} E[g(\tau,\sigma)|\mathcal{F}_n],$$

be minimax and maxmin value processes, respectively. Then we have $\overline{J}_n = \underline{J}_n = J$, say, since $\overline{J}_n = \underline{J}_n$ does not depend upon n in this example. Also, by solving the relation

$$J = \text{mid}(W - c, W + d, E[J]),$$

we have $E[J] = (1 - c + d)/2$, which coincide with $E[U]$. However, the distribution function of J is represented by

$$P(J \le x) = (x - d)I_{[d,(1-c+d)/2)}(x) + (x + c)I_{[(1-c+d)/2,1-c)}(x) + I_{[1-c,\infty)}(x),$$

which is different from that of U, that is, $v(r)$.

5. Acknowledgments

This work was supported by JSPS KAKENHI(21540132).

6. References

Bojdecki, T. (1979). Probability maximizing approach to optimal stopping and its application to a disorder problem. Stochastics, Vol.3, 61–71.

Chow, Y. S.; Robbins, H. & Siegmund, D. (1971). *Great Expectations: The Theory of Optimal Stopping*. Houghton Mifflin, Boston.

DeGroot, M. H. (1970). *Optimal Statistical Decisions*. McGraw Hill, New York.

Denardo, E. V. & Rothblum, U. G. (1979). Optimal stopping, exponential utility, and linear programming. *Math. Programming*, Vol.16, 228–244.

Dynkin, E. B. (1969). Game variant of a problem on optimal stopping. *Soviet Math. Dokl.*, Vol.10, 270–274.

Filar, J. A., Krass, D. & Ross, K. W. (1995). Percentile performance criteria for limiting average Markov decision processes. *IEEE Trans. Automat. Control*, Vol.40, 2-10.

Kadota, Y., Kurano, M. & Yasuda, M. (1996). Utility-optimal stopping in a denumerable Markov chain. *Bull. Informatics and Cybernetics*, Vol.28, 15-21.

Neveu, J. (1975). *Discrete-Parameter Martingales*. North-Holland, New York.

Ohtsubo, Y. (2000). The values in Dynkin stopping problem with th some constraints. *Mathematica Japonica*, Vol.51, 75-81.

Ohtsubo, Y. & Toyonaga, K. (2002). Optimal policy for minimizing risk models in Markov decision processes. *J. Math. Anal. Appl.*, Vol.271, 66-81.

Ohtsubo, Y. (2003). Value iteration methods in risk minimizing stopping problem. *J. Comput. Appl. Math.*, Vol.152, 427-439.

Ohtsubo, Y. (2003). Risk minimization in optimal stopping problem and applications. *J. Operations Research Society of Japan*, Vol.46, 342-352.

Ohtsubo, Y. (2004). Optimal threshold probability in undiscounted Markov decision processes with a target set. *Applied Math. Computation*, Vol.149, 519-532.

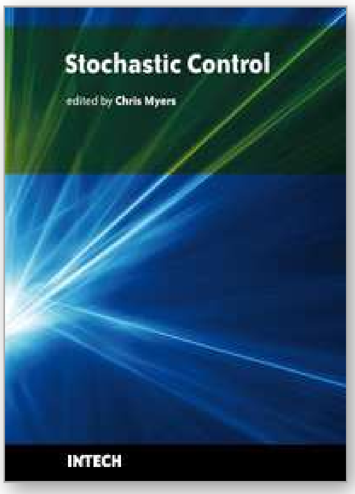
Shiryayev, A. N. (1978). *Optimal Stopping Rules*. Springer, New York.

Uryasev, S. P. (2000). Introduction to theory of probabilistic functions and percentiles (Value-at-Risk). *Probabilistic Constrained Optimization*. Uryasev, S. P., (Ed.), Kluwer Academic Publishers, Dordrecht, pp.1-25.

White, D. J. (1988). Mean, variance and probabilistic criteria in finite Markov decision processes: a review. *J. Optim. Theory Appl.*, Vol.56, 1-29.

White, D. J. (1993). Minimising a threshold probability in discounted Markov decision processes. *J. Math. Anal. Appl.*, Vol.173, 634-646.

Wu, C. & Lin, Y. (1999). Minimizing risk models in Markov decision processes with policies depending on target values. *J. Math. Anal. Appl.* Vol.231, 47-67.



Stochastic Control
Edited by Chris Myers

ISBN 978-953-307-121-3
Hard cover, 650 pages
Publisher Sciyo
Published online 17, August, 2010
Published in print edition August, 2010

Uncertainty presents significant challenges in the reasoning about and controlling of complex dynamical systems. To address this challenge, numerous researchers are developing improved methods for stochastic analysis. This book presents a diverse collection of some of the latest research in this important area. In particular, this book gives an overview of some of the theoretical methods and tools for stochastic analysis, and it presents the applications of these methods to problems in systems theory, science, and economics.