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Application of coloured noise as a driving force in the stochastic differential equations

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Abstract

In this chapter we explore the application of coloured noise as a driving force to a set of stochastic differential equations(SDEs). These stochastic differential equations are sometimes called Random flight models as in A. W. Heemink (1990). They are used for prediction of the dispersion of pollutants in atmosphere or in shallow waters e.g Lake, Rivers etc. Usually the advection and diffusion of pollutants in shallow waters use the well known partial differential equations called Advection diffusion equations(ADEs)R.W.Barber et al. (2005). These are consistent with the stochastic differential equations which are driven by Wiener processes as in P.E. Kloeden et al. (2003). The stochastic differential equations which are driven by Wiener processes are called particle models. When the Kolmogorov's forward partial differential equations(Fokker-Planck equation) is interpreted as an advection diffusion equation, the associated set of stochastic differential equations called particle model are derived and are exactly consistent with the advection-diffusion equation as in A. W. Heemink (1990); W. M. Charles et al. (2009). Still, neither the advection-diffusion equation nor the related traditional particle model accurately takes into account the short term spreading behaviour of particles. This is due to the fact that the driving forces are Wiener processes and these have independent increments as in A. W. Heemink (1990); H.B. Fischer et al. (1979). To improve the behaviour of the model shortly after the deployment of contaminants, a particle model forced by a coloured noise process is developed in this chapter. The use of coloured noise as a driving force unlike Brownian motion, enables to us to take into account the short-term correlated turbulent fluid flow velocity of the particles. Furthermore, it is shown that for long-term simulations of the dispersion of particles, both the particle due to Brownian motion and the particle model due to coloured noise are consistent with the advection-diffusion equation.
Keywords: Brownian motion, stochastic differential equations, traditional particle models, coloured noise force, advection-diffusion equation, Fokker-Planck equation.

1. Introduction

Monte carlo simulation is gaining popularity in areas such as oceanographic, atmospheric as well as electricity spot pricing applications. White noise is often used as an important process in many of these applications which involve some error prediction as in A. W. Heemink

(1990); H.B. Fischer et al. (1979); J. R. Hunter et al. (1993); J.W. Stijnen et al. (2003). In these types of applications usually the deterministic models in the form of partial differential equations are available and employed. The solution is in most cases obtained by discretising the partial differential equations as in G.S. Stelling (1983). Processes such as transport of pollutants and sediments can be described by employing partial differential equations(PDEs). These well known PDEs are called advection diffusion equations. In particular when applied in shallow water e.g River, Lakes and Oceans, such effects of turbulence might be considered. However when this happens, it results into a set of partial differential equations. These complicated set of PDEs are difficult to solve and in most cases not easy to get a closed solution. In this chapter we explore the application coloured noise a a driving force to a set of stochastic differential equations(SDEs). These stochastic differential equations are sometimes called Random flight models. They are used for prediction of the dispersion of pollutants in atmosphere or in shallow waters e.g Lake, Rivers J. R. Hunter et al. (1993); R.W.Barber et al. (2005). Usually the advection and diffusion of pollutants in shallow waters use the well known partial differential equations called Advection diffusion equations(ADEs). These are consistent with the stochastic differential equations which are driven by Wiener processes as in C.W. Gardiner (2004); P.E. Kloeden et al. (2003). The stochastic differential equations which are driven by Wiener processes are called particle models. When the Kolmogorov's forward partial differential equations(Fokker-Planck equation) is interpreted as an advection diffusion equation, the associated with this set of stochastic differential equations called particle model are derived and are exactly consistent with the advection-diffusion equation as in W. M. Charles et al. (2009). Still, neither the advection-diffusion equation nor the related traditional particle model accurately takes into account the short term spreading behaviour of particles. This is due to the fact that the driving forces are Wiener processes and these have independent increment. To improve the behaviour of the model shortly after the deployment of contaminants, a particle model forced by a coloured noise process is developed in this article. The use of coloured noise as a driving force unlike Brownian motion, enables to us to take into account the short-term correlated turbulent fluid flow velocity of the particles. Furthermore, it is shown that for long-term simulations of the dispersion of particles, both the particle due to Brownian motion and the particle model due to coloured noise are consistent with the advection-diffusion equation.

To improve the behaviour of the model shortly after the deployment of contaminants, a random flight model forced by a coloured noise process are often used. The scheme in Figure 1, shows that for long term simulation both models, advection diffusion equation and the random flight models have no difference, such situation better to use the well known ADE. The use of coloured noise as a driving force unlike Brownian motion, enables to us to take into account only the short-term correlated turbulent fluid flow velocity of the particles as in A. W. Heemink (1990); W. M. Charles et al. (2009). An exponentially coloured noise process can also be used to mimic well the behaviour of electricity spot prices in the electricity market. Furthermore, when the stochastic numerical models are driven by the white noise, in most cases their order of accuracy is reduced. Such models consider that particles move according to a simple random walk and consequently have independent increment as in A.H. Jazwinski (1970); D.J. Thomson (1987). The reduction of the order of convergence happens because white noise is nowhere differentiable. However, one can develop a stochastic numerical scheme and avoid the reduction of the order of convergence if the coloured noise is employed as a driving force as in A. W. Heemink (1990); J.W. Stijnen et al. (2003); R.W.Barber et al. (2005); P.S. Addison et al. (1997).

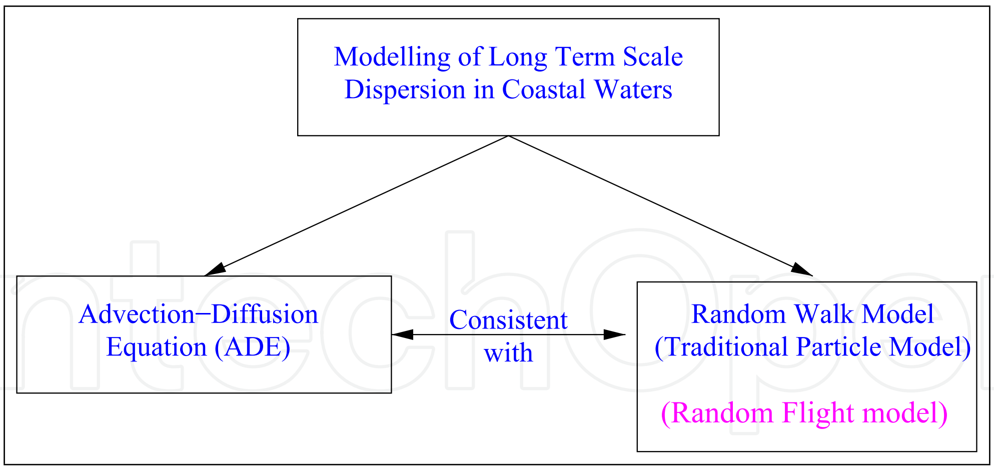

Fig. 1. A schematic diagram showing that for $t >> T_L$ both the ADEs and Random flight models are consistent

The application of coloured noise as a driving force to improve the model prediction of the dispersion of pollutants soon after deployment is discussed in this chapter. For it is well-known that the advection-diffusion equation describes the dispersion of particles in turbulent fluid flow accurately if the diffusing cloud of contaminants has been in the flow longer than a certain Lagrangian time scale and has spread to cover a distance that is larger in size than the largest scale of the turbulent fluid flow as in H.B. Fischer et al. (1979). The Lagrangian time scale (T_L) is a measure of how long it takes before a particle loses memory of its initial turbulent velocity. therefore, both the particle model which is driven by Brownian force and the advection-diffusion model are unable to accurately describe the short time scale correlated behaviour which is available in real turbulent flows at sub-Lagrangian time. Thus, a random flight model have been developed for any length of the coloured noise. This way, the particle model takes correctly into account the diffusion processes over short time scales when the eddy(turbulent) diffusion is less than the molecular diffusion. The inclusion of several parameters in the coloured noise process allows for a better match between the auto-covariance of the model and the underlying physical processes.

2. Coloured noise processes

In this part coloured noise forces are introduced and represent the stochastic velocities of the particles, induced by turbulent fluid flow. It is assumed that this turbulence is isotropic and that the coloured noise processes are stationary and completely described by their zero mean and Lagrangian auto covariance functionH.M. Taylor et al. (1998); W. M. Charles et al. (2009).

2.1 The scalar exponential coloured noise process

The exponentially coloured noise are represented by a linear stochastic differential equation. The exponential coloured noise represent the velocity velocity of the particle;

$$du_1(t) = -\frac{1}{T_L}u_1(t)dt + \alpha_1 dW(t). \tag{1}$$

$$u_1(t) = u_0 e^{\frac{-t}{T_L}} + \alpha_1 \int_0^t e^{-\frac{(t-s)}{T_L}} dW(s) \tag{2}$$

where u_1 is the particle's velocity, $\alpha_1 > 0$ is constant, and T_L is a Lagrangian time scale. For $t > s$ it can be shown as in A.H. Jazwinski (1970), that the scalar exponential coloured noise process in Eqn. (2) has mean, variance and Lagrangian auto-covariance of respectively,

$$\mathbb{E}[u_1(t)] = u_0 e^{\frac{-t}{T_L}},\ \mathrm{Var}[u_1(t)] = \frac{\alpha_1^2 T_L}{2}\left(1 - e^{-\frac{2t}{T_L}}\right),$$
$$\mathrm{Cov}[u_1(t), u_1(s)] = \frac{\alpha_1^2 T_L}{2} e^{-\frac{|t-s|}{T_L}}. \tag{3}$$

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$$\mathrm{Cov}[u_1(t), u_1(s)] = \frac{\alpha_1^2 T_L}{2} e^{-\frac{|t-s|}{T_L}}. \tag{4}$$

2.2 The general vector coloured noise force

The general vector form of a linear stochastic differential equation for coloured noise processes as in A.H. Jazwinski (1970); H.M. Taylor et al. (1998) is given by

$$d\mathbf{u}(t) \quad = \quad \mathrm{F}\mathbf{u}(t)dt + \mathrm{G}(t)dW(t),\ \ d\mathbf{v}(t) = \mathrm{F}\mathbf{v}(t)dt + \mathrm{G}(t)dW(t). \tag{5}$$

Where $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vectors of length n, F and G are $n \times n$ respectively $n \times m$ matrix functions in time and $\{W(t); t \geq 0\}$ is an m-vector Brownian process with $\mathbb{E}[dW(t)dW(t)^{\mathrm{T}}] = Q(t)dt$. In this chapter, a special case of the Ornstein-Uhlenbeck process C.W. Gardiner (2004); H.M. Taylor et al. (1998) is extended and repeatedly integrate it to obtain the coloured noise forcing along the x and y-directions:

$$\begin{array}{lclllcl}
du_1(t) & = & -\frac{1}{T_L}u_1(t)dt + \alpha_1 dW(t), & & dv_1(t) & = & -\frac{1}{T_L}v_1(t)dt + \alpha_1 dW(t) \\
du_2(t) & = & -\frac{1}{T_L}u_2(t)dt + \frac{1}{T_L}\alpha_2 u_1(t)dt, & & dv_2(t) & = & -\frac{1}{T_L}v_2(t)dt + \frac{1}{T_L}\alpha_2 v_1(t)dt \\
du_3(t) & = & -\frac{1}{T_L}u_3(t)dt + \frac{1}{T_L}\alpha_3 u_2(t)dt, & & dv_3(t) & = & -\frac{1}{T_L}v_3(t)dt + \frac{1}{T_L}\alpha_3 v_2(t)dt \\
du_4(t) & = & -\frac{1}{T_L}u_4(t)dt + \frac{1}{T_L}\alpha_4 u_3(t)dt, & & dv_4(t) & = & -\frac{1}{T_L}v_4(t)dt + \frac{1}{T_L}\alpha_4 v_3(t)dt \\
du_5(t) & = & -\frac{1}{T_L}u_5(t)dt + \frac{1}{T_L}\alpha_5 u_4(t)dt, & & dv_5(t) & = & -\frac{1}{T_L}v_5(t)dt + \frac{1}{T_L}\alpha_5 v_4(t)dt \\
du_6(t) & = & -\frac{1}{T_L}u_6(t)dt + \frac{1}{T_L}\alpha_6 u_5(t)dt, & & dv_6(t) & = & -\frac{1}{T_L}v_6(t)dt + \frac{1}{T_L}\alpha_6 v_5(t)dt \\
\vdots & = & \vdots & & \vdots & = & \vdots \\
du_n(t) & = & -\frac{1}{T_L}u_n(t)dt + \frac{1}{T_L}\alpha_n u_{n-1}(t)dt, & & dv_n(t) & = & -\frac{1}{T_L}v_n(t)dt + \frac{1}{T_L}\alpha_n v_{n-1}(t)dt
\end{array} \tag{6}$$

As you keep increasing the length of the coloured noise, an auto-covariance of the velocity processes is modelled more realistically to encompasses the characteristics of an isotropic homogeneous turbulent fluid flow.

Figure 2 in an example of Wiener path and that of a coloured noise process. The sample path of the coloured noise are smoother that that of Wiener process.

The vector Langevin equation (6) generates a stationary, zero-mean, correlated Gaussian process denoted by $(u_n(t), v_n(t))$. The Lagrangian time scale $\mathrm{T_L}$ indicates the time over which the process remains significantly correlated in time. The linear system in eqn.(6), is the same in

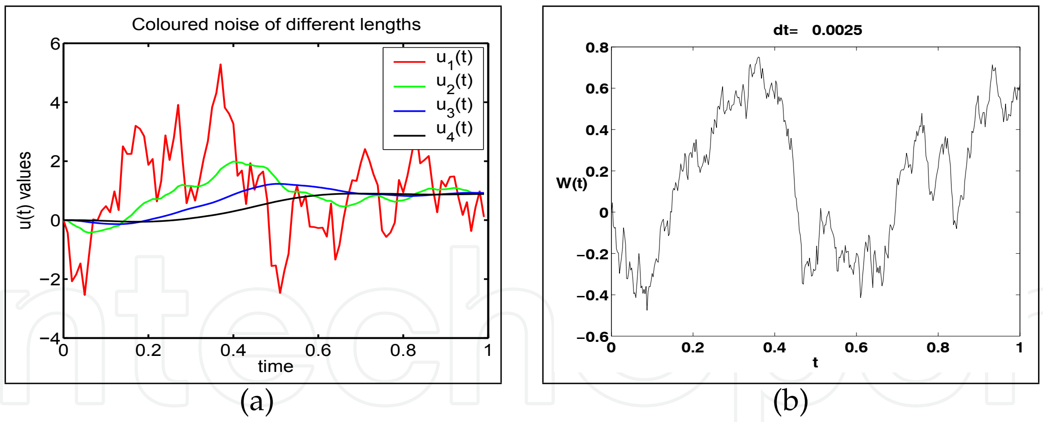


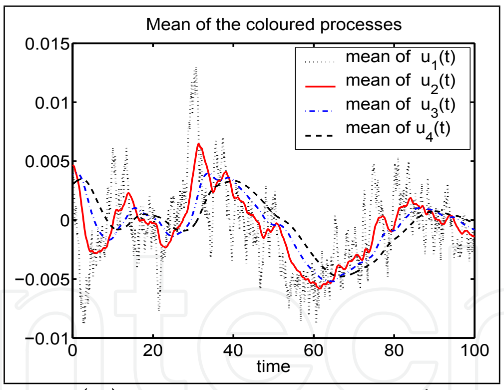
Fig. 2. Sample paths of coloured noise (a) and sample path of Wiener process (b)

the Itô and the Stratonovich sense because the diffusion function is not a function of state but only of time. In order to get more accurate results the stochastic differential equation driven by the coloured is integrated. by the Heun scheme (see e.g., G.N. Milstein (1995); J.W. Stijnen et al. (2003); P.E. Kloeden et al. (2003)).

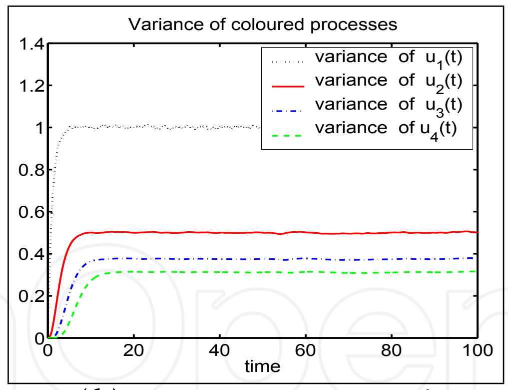
The main purpose of this chapter is the application of coloured noise forcing in the dispersion of a cloud of contaminants so as to improve the short term behaviour of the model while leaving the long term behaviour unchanged. Being the central part of the model, it is important to study the properties of coloured noise processes in more detail. Coloured noise is a Gaussian process and it is well known that these processes can be completely described by their mean and covariance functions see L. Arnold (1974). From eqn.(2) and from Figure 3(a), it is easily seen that the mean approaches zero throughout and therefore requires little attention. The covariance, however, depends not only on time but also on the initial values of $u_n(0)$ and $v_n(0)$. This immediately gives rise to the question of how to actually choose or determine these values. Let's consider the covariance matrix of the stationary process $\mathbf{u}$ in the stochastic differential equations of the form (5). It is known (see e.g.,A.H. Jazwinski (1970)) that covariance function can now be described by

$$\frac{dP}{dt} = FP + PF^T + GQG^T. \tag{7}$$

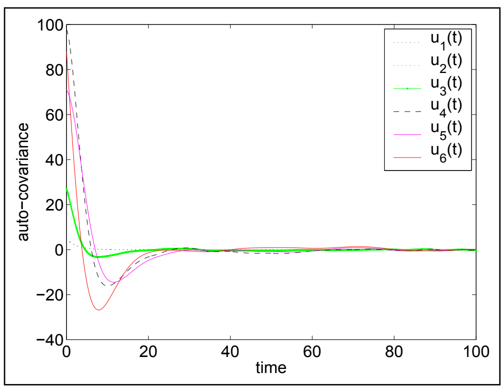
The equation (7) can be equated zero so as to find the steady state covariance matrix $\bar{P}$ which will then be used to generate instances of coloured noise processes. Sampling of instances of u vector by using a steady state matrix, ensures that the process is sampled at its stationary phase thus removing any artefacts due to a certain choice of start values that would otherwise be used. The auto-covariance is depicted in Figure 3(c). Note that the behaviour of a physical process in this case depends on the parameters in the Lagrangian auto-covariance. Of course short term diffusion behaviour is controlled by the auto-covariance function. This provides room for the choice of parameters e.g.,$\alpha_1, \alpha_2 \cdots$. The mean, variance and the auto-covariance are not stationary for a finite time t but as $t \to \infty$,they approach the limiting stationary distribution values as shown in Figure 3(a)–(c).



(a) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$



(b) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$



(c) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$

Fig. 3. (a)Shows that the mean goes to zero, while (b)-(c) shows that the variance and auto-covariance of coloured noise processes started from non-stationary to stationary state

2.3 The particle model forced by coloured noise

The prediction of the dispersion of pollutants in shallow waters are modeled by the random flight which is driven by coloured as in A. W. Heemink (1990). In this work, an extension to the work by A. W. Heemink (1990) has been done by generalising the cloured noise to any length that is, to $(u_n(t), v_n(t))$. The coloured noise processes stand for the velocity of the particle at time t in respectively the x and y directions. This way the Lagrangian auto-covariance processes can be modelled more realistically by taking into account the characteristics of the turbulent fluid flow for $t \ll T_L$. By using the following set of equations the random flight model remains consistent with the advection-diffusion equation for $t >> T_L$ while modelling realistically the short term correlation of the turbulent fluid flows. In this application, unlike in W. M. Charles et al. (2005), Longer length of the coloured noise have been chosen, that is $n = 6$ and more experiments are carried out in the whirl pool ideal domain for simulations of the advection and diffusion of pollutants in shallow waters. Thus the following coloured noise are used.

$$
\begin{aligned}
du_1(t) &= -\tfrac{1}{T_L}u_1(t)dt + \alpha_1 dW(t), & dv_1(t) &= -\tfrac{1}{T_L}v_1(t)dt + \alpha_1 dW(t)\\
du_2(t) &= -\tfrac{1}{T_L}u_2(t)dt + \tfrac{1}{T_L}\alpha_2 u_1(t)dt, & dv_2(t) &= -\tfrac{1}{T_L}v_2(t)dt + \tfrac{1}{T_L}\alpha_2 v_1(t)dt\\
du_3(t) &= -\tfrac{1}{T_L}u_3(t)dt + \tfrac{1}{T_L}\alpha_3 u_2(t)dt, & dv_3(t) &= -\tfrac{1}{T_L}v_3(t)dt + \tfrac{1}{T_L}\alpha_3 v_2(t)dt\\
du_4(t) &= -\tfrac{1}{T_L}u_4(t)dt + \tfrac{1}{T_L}\alpha_4 u_3(t)dt, & dv_4(t) &= -\tfrac{1}{T_L}v_4(t)dt + \tfrac{1}{T_L}\alpha_4 v_3(t)dt\\
du_5(t) &= -\tfrac{1}{T_L}u_5(t)dt + \tfrac{1}{T_L}\alpha_5 u_4(t)dt, & dv_5(t) &= -\tfrac{1}{T_L}v_5(t)dt + \tfrac{1}{T_L}\alpha_5 v_4(t)dt\\
du_6(t) &= -\tfrac{1}{T_L}u_6(t)dt + \tfrac{1}{T_L}\alpha_6 u_5(t)dt, & dv_6(t) &= -\tfrac{1}{T_L}v_6(t)dt + \tfrac{1}{T_L}\alpha_6 v_5(t)dt
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
dX(t) &= \left[U + \sigma u_6(t) + (\tfrac{\partial H}{\partial x}D)/H + \tfrac{\partial D}{\partial x}\right] dt\\
dY(t) &= \left[V + \sigma v_6(t) + (\tfrac{\partial H}{\partial y}D)/H + \tfrac{\partial D}{\partial y}\right] dt.
\end{aligned}
\tag{9}
$$

These systems of vector equations are Markovian, this set of equations are referred to as the random flight model. The random flight model(8)–(9) is integrated for many different particles. Note that at the start of the simulation all particles have initial Gaussian velocities $(u_6(0), v_6(0))$ with zero mean and variance that agrees with covariance matrix $\bar{P}$ at a steady state. For instance in this chapter, the following covariance matrix was obtained when the parameters shown in Table 1 were used in the simulation;

$$
\begin{bmatrix}
25000 & 17500 & 2625 & 26.25 & 15.75 & 11.025\\
17500 & 24500 & 5512 & 73.50 & 55.125 & 46.305\\
2625 & 5512.5 & 1653.75 & 27.5625 & 24.806250 & 24.310125\\
26.25 & 73.5 & 27.5625 & 0.551250 & 0.578812 & 0.648270\\
15.75 & 55.125 & 24.80625 & 0.578812 & 0.694575 & 0.875164\\
11.025 & 46.305 & 24.310125 & 0.648270 & 0.875164 & 1.22523
\end{bmatrix}
$$

3. The spreading behaviour of a cloud of contaminants

The characteristics of a spreading cloud of contaminants due to Brownian motion and coloured noise processes are discussed in the following sections.

3.1 Long term spreading behaviour of clouds of particles due Brownian motion force

Consider, the following 1 dimensional stochastic differential equation in the Itô sense

$$dX(t) \overset{\text{Itô}}{=} f(t, X_t)dt + g(t, X_t)dW(t) \tag{10}$$

where $f(t, X_t)$ is the drift coefficient function and where $g(t, X_t)$ is the diffusion coefficient function. If it assumed that there is no drift term in eqn.(10) that is, $f(X(t), t) = 0$, gives

$$g(X(t), t) = \sqrt{2D}.$$

It follows that,

$$dX(t) \overset{\text{Itô}}{=} \sqrt{2D}dW(t). \tag{11}$$

By applying following theorem which is found in H.M. Taylor et al. (1998), that for any continuous function the following theorem is applied.

Theorem 1. *Let $g(x)$ be continuous function and $\{W(t), t \geq 0\}$ be the standard Brownian motion process H.M. Taylor et al. (1998). For each $t > 0$, there exits a random variable*

$$\mathcal{F}(g) = \int_0^t g(x) dW(x),$$

which is the limiting of approximating sums

$$\mathcal{F}_n(g) = \sum_{k=1}^{2^n} g(\frac{k}{2^n}t)[W(\frac{k}{2^n}t) - W(\frac{k-1}{2^n}t)],$$

as $n \to \infty$. The random variable $\mathcal{F}(g)$ is normally distributed with mean zero and variance

$$\mathrm{Var}[\mathcal{F}(g)] = \int_0^t g^2(u) du,$$

if $f(x)$ is another continuous function of x then $\mathcal{F}(f)$ and $\mathcal{F}(g)$ have a joint normal distribution with covariance

$$\mathbb{E}[\mathcal{F}(f)\mathcal{F}(g)] = \int_0^t f(x)g(x)dx.$$

to eqn.(11), it can be shown that the variance of a cloud of contaminants grows linearly with time:

$$\mathrm{Var}[X(t)] \overset{\text{Itô}}{=} 2Dt + \text{constant}. \tag{12}$$

For more detailed information as well as the proof of this theorem, the reader is referred to H.M. Taylor et al. (1998) for example.

3.2 Long term spreading behaviour of clouds of contaminants subject to coloured noise forcing

As discussed in earlier, where for example, the first an exponential coloured $u_1(t)$ from eqn (2) is used as forcing coloured noise, if it is assumed that there is no background flow, the position of a particle at time t is given by

$$dX(t) \quad = \quad \sigma u_1(t)dt, \Longrightarrow X(t) = X(0) + \sigma \int_0^t u_1(m)dm. \tag{13}$$

For simplicity, yet without loss of generality, let $X(0) = u_i(0) = 0$, for $i = 1, 2, \cdots, n$. Now, eqn., (2) leads to $u_1(m) = \alpha_1 \int_0^m e^{-\frac{1}{T_L}(m-k)} dW(k)$, and consequently,

$$X(t) \quad \overset{\text{Itô}}{=} \quad \sigma\alpha_1 T_L \int_0^t (1 - e^{-\frac{1}{T_L}(t-k)}) dW(k). \tag{14}$$

Using Theorem 1, the position of a particle at time t is normally distributed with zero mean and variance:

$$\frac{\mathrm{Var}[X(t)]}{t} = \sigma^2 \alpha_1^2 T_L^2 \left[1 - \frac{2T_L}{t}(1 - e^{\frac{-t}{T_L}}) + \frac{T_L}{2t}(1 - e^{\frac{-2t}{T_L}})\right].$$

Thus, a position of a particle observed over a long time span as modelled by the coloured noise process $u_1(t)$ behaves much like the one driven by Brownian motion with variance parameter

$\sigma^2\alpha_1^2 T_L^2$. Hence, the dispersion coefficient is related to variance parameters $\sigma^2\alpha_1^2 T_L^2 = 2D$. Clarification are done by considering eqn.(14), where the second part is $u_1(t)$ itself;

$$X(t) = \sigma T_L \left[\alpha_1 W(t) - u_1(t)\right], \text{ where } u_1(t) = \alpha_1 \int_0^t e^{-\frac{1}{T_L}(t-k)} dW(k)$$

Let us now rescale the position process in order to better observe the changes over large time spans. By doing so, for $N > 0$, yields,

$$X_N(t) = \frac{1}{\sqrt{N}} X(Nt) \;= \sigma T_L \left[\alpha_1 \tilde{W}(t) + \frac{1}{\sqrt{N}} u_1(t)\right], \tag{15}$$

where $\tilde{B}(t) = \frac{W(Nt)}{\sqrt{N}}$ remains a standard Brownian motion process. For sufficiently large N it becomes clear that eqn.(15) behaves like Brownian motion as in H.M. Taylor et al. (1998); W. M. Charles et al. (2009):

$$X_N(t) \quad \approx \sigma\alpha_1 T_L \tilde{W}(t).$$

3.3 The analysis of short term spreading behaviour of a cloud of contaminants

The analysis of the coloured noise processes usually starts with a scalar coloured noise, it can be shown using eqn.(4) that

$$\mathrm{Cov}[u_{t+\tau} u_t] \quad = \quad \mathbb{E}[u_{t+\tau} u_t] = \mathbb{E}[v_{t+\tau} v_t] = \frac{1}{2}\alpha_1^2 T_L e^{\frac{-|\tau|}{T_L}} \tag{16}$$

From equation (16), It follows that,

$$\mathbb{E}[u_\tau u_s] = \frac{1}{2}\alpha_1^2 T_L e^{\frac{-|\tau - s|}{T_L}}$$

$$\mathrm{Var}[X_t] = \sigma^2 \int_0^t \int_0^t \frac{1}{2}\alpha_1^2 T_L e^{\frac{-(\tau - s)}{T_L}} d\tau ds \tag{17}$$

The integration of equation (17) can easily be yielded by separately considering the regions $\tau < s$ and $\tau > s$, and it can be shown that

$$\begin{aligned}
\mathrm{Var}[X_t] \quad &= \quad \sigma^2\alpha_1^2 T_L^3 \left(\frac{t^2}{2T_L^2} - \frac{t^3}{6T_L^3} \cdots\right) \\
&= \quad \frac{\sigma^2\alpha_1^2 T_L t^2}{2} - \frac{\sigma^2\alpha_1^2 t^3}{6} + \cdots
\end{aligned} \tag{18}$$

Since the short time analysis, eqn. (18) are of interest in this section and is considered only for very small values of t in a sense that for $t \ll T_L$ the variance of a cloud of particles shortly after deployment is then given by the following equation:

$$\mathrm{Var}[X_t] = \frac{1}{2}\sigma^2\alpha_1^2 T_L t^2 \tag{19}$$

With the constant dispersion coefficient $D = \frac{1}{2}\sigma^2\alpha_1^2 T_L^2$, the variance of the cloud of particles, therefore initially grows with the square of time:

$$\mathrm{Var}[X(t)] = \frac{D}{T_L} t^2 \tag{20}$$

3.4 The general long term behaviour of a cloud of contaminants due to coloured noise

It is assumed that there is no flow in the model and therefore have

$$dX(t) = \sigma u_1(t)dt \longrightarrow X(t) = \int_0^t \sigma u_1(s)ds,\ \ u_1(s) = \alpha_1 \int_0^s e^{-\frac{1}{T_L}(s-k)} dW(k),$$

$$X(t) = \left(\frac{1}{T_L}\right)^0 \sigma\alpha_1 \int_0^t \int_0^{t-k} e^{-\frac{1}{T_L}(s-k)} \frac{(s-k)^0}{0!} dsdW(k),\ X(0) = 0$$

It can then be shown that

$$u_2(s) \quad = \tfrac{1}{T_L}\alpha_1\alpha_2 \int_0^s e^{-\frac{1}{T_L}(s-k)}(s-k)dW(k), \tag{21}$$

where $0 < m < s < t$. Since $k < s$, the position of a particle due to coloured noise force eqn.(21) is given by

$$X(t) = \left(\frac{1}{T_L}\right)^1 \sigma\alpha_1\alpha_2 \int_0^t \int_0^{t-k} e^{-\frac{1}{T_L}(s-k)} \frac{(s-k)^1}{1!} dsdW(k).$$

In general, a position due to $u_n(t)$ force is: $X(t) = \int_0^t \sigma u_n(s)ds$, and it follows that

$$X(t) \stackrel{\text{Itô}}{=} \left(\frac{1}{T_L}\right)^{n-1} \sigma \prod_{i=1}^n \alpha_i \int_0^t \left[\int_0^{t-k} e^{-\frac{1}{T_L}(s-k)} \frac{(s-k)^{n-1}}{(n-1)!}\right] dsdW(k) \tag{22}$$

Careful manipulation using integration by parts of the integral within the square brackets of eqn. (22), yields

$$X(t) \stackrel{\text{Itô}}{=} (T_L)^n \left(\frac{1}{T_L}\right)^{n-1} \sigma \prod_{i=1}^n \alpha_i \int_0^t [1+\ldots]\, dW(k),\ \text{ for } n \geq 1. \tag{23}$$

Finally, with the aid of Theorem 1 whose proof is found in H.M. Taylor et al. (1998), the variance of a cloud of contaminants can be computed as described in the sections above. The derivation of velocity $v_n(t)$ of the particle along the y direction proceeds completely analogously. Let us now compute the variance of the general equations for position given by eqn.(23)

$$\text{Var}[X(t)] \quad = \sigma^2 (T_L)^2 \textstyle\prod_{i=1}^n \alpha_i^2 \int_0^t [1-\ldots]^2\, dk \tag{24}$$

For $\sigma > 0$, $\alpha_i > 0$, and $T_L > 0$, the process again behaves like a Brownian process with variance parameters $T_L^2\sigma^2 \prod_{i=1}^n \alpha_i^2$ as $t \to \infty$. Thus the appropriate diffusion coefficient from eqn.(12)is equals $D = \frac{\sigma^2 T_L^2 \prod_{i=1}^n \alpha_i^2}{2}$. This relation is important because it gives a criterion for various choices of parameters α_i , $i = 1, \cdots, n$, $T_L > 0$. In a simulation the constant dispersion coefficient D often is specified whereas σ must be solved in terms of the other parameters. In the following section we introduce the two dimensional particle model as in W. M. Charles et al. (2009). This model will be used as a comparison with the random flight model during the simulation of the dispersion of pollutants an ideal domain known as whirl pool.

4. Particle model due to Brownian motion force for dispersion of pollutants in shallow waters

The position of particles in water at time t, is designated by $(X(t), Y(t))$. Different random locations of the particle are described with the aid of stochastic differential equation. The integration of the movements of the particle in water is done in two steps. A deterministic step consisting of velocity field of water and a random step known as diffusion modelled by the stochastic process A. W. Heemink (1990);

$$dX(t) \overset{\text{Itô}}{=} \left[U + \frac{D}{H}\frac{\partial H}{\partial x} + \frac{\partial D}{\partial x}\right] dt + \sqrt{2D}dW_1(t), \;\; X(0) = x_0 \tag{25}$$

$$dY(t) \overset{\text{Itô}}{=} \left[V + \frac{D}{H}\frac{\partial H}{\partial y} + \frac{\partial D}{\partial y}\right] dt + \sqrt{2D}dW_2(t), \;\; Y(0) = y_0. \tag{26}$$

Here D is the dispersion coefficient in m^2/s; $U(x, y, t),\ V(x, y, t)$ are the averaged flow velocities (m/s) in respectively x, y directions; $H(x, y, t)$ is the total depth in m at location (x, y), and $dW(t)$ is a Brownian motion with mean $(0, 0)^T$ and $\mathbb{E}[dW_1(t)dW_2(t)^{\mathrm{T}}] = Idt$ where I is a 2×2 identity matrixP.E. Kloeden et al. (2003). Note that the advective part of the particle model eqns.(25)–(26) is not only containing the averaged water flow velocities but also spatial variations of the diffusion coefficient and the averaged depth. This correction term makes sure that particles are not allowed to be accumulated in regions of low diffusivity as demonstrated by (see e.g., J. R. Hunter et al. (1993); R.W.Barber et al. (2005)). At closed boundaries particle bending is done by halving the time step sizes until the particle no longer crosses closed boundary. As a result there is no loss of mass through such boundaries. The position $(X(t), Y(t))$ process is Markovian and the evolution of its probability density function $(p(x, y, t))$, is described by an advection-diffusion type of the partial differential equation known as the Fokker-Planck equation (see e.g.,A.H. Jazwinski (1970))

5. Discrete version of the particle model driven by Brownian motion

Analytical solutions of stochastic differential equations do not always exist due to their complexity and nonlinearity. Therefore, stochastic numerical integration schemes are often applied as in G.N. Milstein (1995); J.W. Stijnen et al. (2003). An example of a numerical scheme is the Euler scheme which, although not optimal in terms of order of convergence, is easy to implement and requires only $O(\Delta t)$ in the weak sense P.E. Kloeden et al. (2003). Here the time interval $[t_0, T]$ is discretised as $t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T$, with $\Delta(t_k) = t_{k+1} - t_k$, $\Delta W(t_k) = W(t_{k+1}) - W(t_k)$, for $k = 0, 1, \cdots n$.

$$\bar{X}(t_{k+1}) = \bar{X}(t_k) + \left[U + (\tfrac{\partial H}{\partial x}D)/H + \tfrac{\partial D}{\partial x}\right] \Delta(t_k) + \sqrt{2D}\Delta W(t_k) \tag{27}$$

$$\bar{Y}(t_{k+1}) = \bar{Y}(t_k) + \left[V + (\tfrac{\partial H}{\partial y}D)/H + \tfrac{\partial D}{\partial y}\right] \Delta(t_k) + \sqrt{2D}\Delta W(t_k) \tag{28}$$

Where $\bar{X}(t_{k+1})$ and $\bar{Y}(t_{k+1})$ are the numerical approximations of the $X(t_{k+1})$ and $Y(t_{k+1})$ positions respectively due to the traditional particle model. The noise increments $\Delta W(t_k)$ are independent and normally distributed $\mathrm{N}\left(0, \Delta(t_k)\right)$ random variables which can be generated using e.g., pseudo-random number generators. The domain information consisting of flow velocities and depth is computed using a hydrodynamic model known as WAQUA see G.S. Stelling (1983). The flow averaged fields are only available on grid points of a rectangularly discretised grid and therefore, interpolation methods are usually used to approximate the values at other positions.

5.1 Boundaries

Numerical schemes such as the Euler scheme often show very poor convergence behaviour G.N. Milstein (1995); P.E. Kloeden et al. (2003). This implies that, in order to get accurate results, small time steps are needed thus requiring much computation. Another problem with the Euler (or any other numerical scheme) is its undesirable behaviour in the vicinity of boundaries; a time step that is too large may result in particles unintentionally crossing boundaries. To tackle this problem two types of boundaries are prescribed. Closed boundaries representing boundaries intrinsic to the domain, and open boundaries which arise from the modeller's decision to artificially limit the domain at that location. Besides these boundary types, the is of what what happens if, during integration, a particle crosses one of these two borders is also considered as in J.W. Stijnen et al. (2003); W. M. Charles et al. (2009);

- In case an open boundary is crossed by a particle, the particle remains in the sea but is now outside the scope of the model and is therefore removed;
- In case a closed boundary is crossed by a particle during the advective step of integration, the step taken is cancelled and the time step halved until the boundary is no longer crossed. However, because of the halving, say n times, the integration time is reduced to $2^{-n}\Delta t$, leaving a remaining $(1-2^{-n})\Delta t$ integration time. This means at least another $2^n - 1$ steps need to be taken at the new integration step in order to complete the full time-step Δt. This way, shear along the coastline is modelled;
- If a closed boundary is crossed during the diffusive part of integration, the step size halving procedure described above is maintained with the modification that in addition to the position, the white noise process is also restored to its state prior to the abandoned integration step. Again the process of halving the time step and continuing integration is repeated until no boundaries are crossed and the full Δt time step has been integrated.

6. Numerical Experiments

Before applying both the traditional model(25)–(26) and the part model forced by coloured noise(8)–(9) to a real life pollution problem, A whirl pool have been created as domain for test problem. In this case a whirl pool domain with flow field and a constant total depth of 25 metres is created. In order to compare the spreading behaviour of a cloud of contaminants some experiments using both particle models have been carried out. The table 1 below summarises the simulation parameters that have been used in the experiments:

Summary of the simulation parameters of particle for pollutants dispersion in shallow waters

Whirl pool	Unit	Value
# of steps	-	89999
Δt	s	86.4
Particles	-	5000
$\alpha_1, \alpha_2, \alpha_3$	-	$\alpha_1 = 1, \alpha_2 = 1.4, \alpha_3 = 0.3$
$\alpha_4, \alpha_5, \alpha_6$	-	$\alpha_4 = 0.02, \alpha_5 = 1.2, \alpha_6 = 1.4$
Tracks	-	5
Grid offset	m	$(-20800, -20800)$
Grid size	-	105×105
Cell size	m	400×400
Init. point	m	$(-10000, 14899)$
D	m/s^2	3
T_L	s	50000

Table 1. The simulation parameters of the particle model for the dispersion of pollutants in shallow waters.

From now onwards in this section Brownian motion is denoted by BM and coloured noise by CN. A bunch of 5000 particles are released at the location $(-10000, 14899)$, the simulation starts at time $t_0 = 0$ in the whirl pool domain. The scattering of a cloud of contaminants due to coloured noise or Brownian motions forces is followed at a specified time steps after release. Generally a large number of particles are used P.S. Addison et al. (1997) in numerical simulations. The simulations parameters that have been used for simulations of advection and diffusion of pollutants in shallow waters in this article are summarised in the Table 1. The results in the Figures 4(a)-(b) show that a cloud of 5000 particles have been deployed, while Figure 4(c)-(d) shows that 5000 have spread to cover a certain distance 52 days later for random flight model by CN and particle model by BM respectively. Whereas Figure5(a)-(b) realisations of marked 5 tracks by CN and BM noised respectively while Figure5(c)-(d) show a realisation of a single same marked particle for all the simulation period of 89999 time steps each of 86.4s.

In this chapter a series of experiments are carried out in a stationary homogeneous turbulent flow with zero mean velocity. The Lagrangian time scale as T_L is introduced in the models. Futhermore, an experiment was carried in the empty domain as in W. M. Charles et al. (2009) using random flight model as well as the traditional particle model so as to show the differences between the small scale fluctuations and their similarity in the long scale fluctuations. The simulation of the spreading of a cloud of $20,000$ particles is tracked in an empty domain and its variance is computed. It has been shown that once the particles have been in the flow longer than the time scale T_L, the variance of the spreading cloud grows linearly with time similar to the behaviour of the advection-diffusion equation. Before that time, the variance grows with the square of time (quadratically), creating two different zones of diffusion see Figure 6 as in W. M. Charles et al. (2009). In Section 3.4 it is suggested that for $t \gg TL$ a turbulent mixing coefficient similar to constant dispersion coefficient D such that $D = \frac{\sigma^2 T_L^2 \prod \alpha_i^2}{2}$ can be defined.

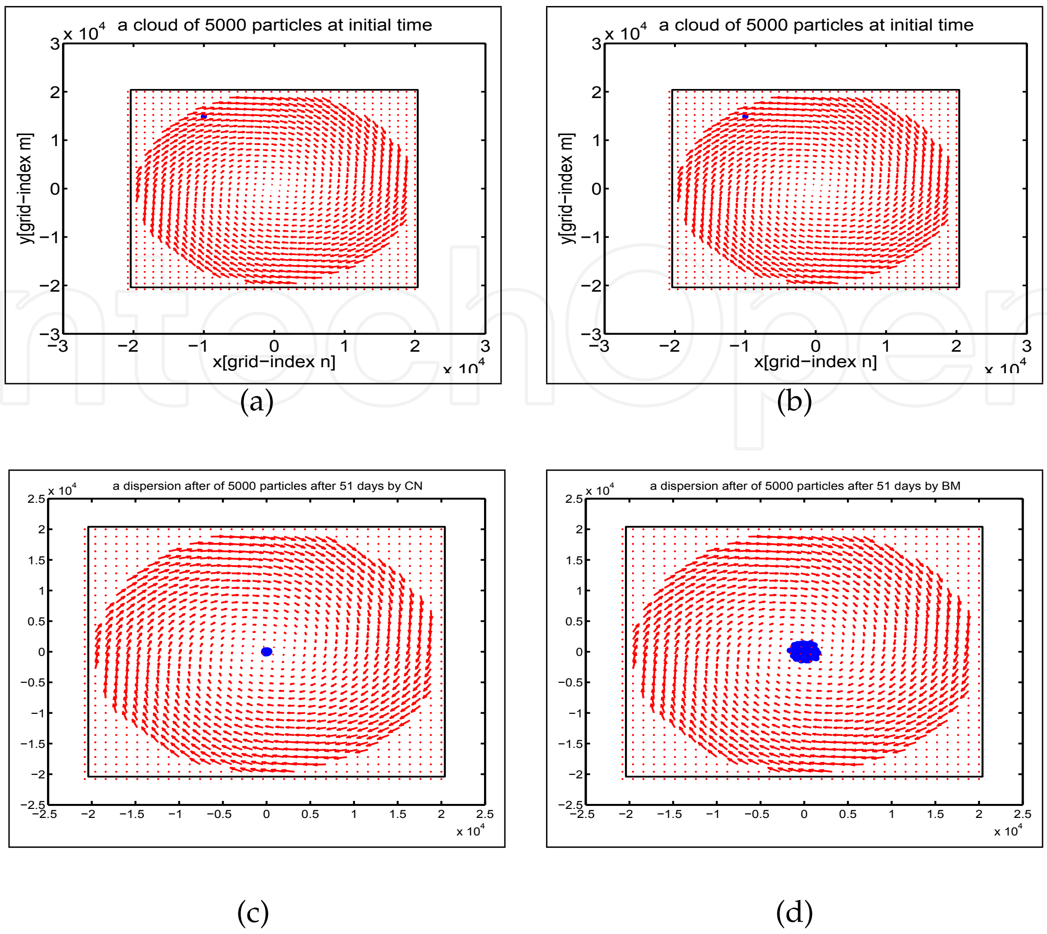


Fig. 4. Dispersion of a cloud of 5000 particles released in the idealised whirl pool domain. (a) Due to coloured noise for $t << T_L$, (b) Due to Brownian motion noise for $t << T_L$, (c) Due to coloured noise for $t >> T_L$, (d) Due to Brownian motion noise for $t >> T_L$.

7. Conclusions

The results obtained in this work suggest that coloured noised can be used to improve the prediction of the dispersion of pollutants. This is possible when a short time correlation is considered which is that case in most cases. Thus, random flight model can provide the modeller with an enhanced tool for the short term simulation of the pollutants by providing more flexibility to account for correlated physical processes of diffusion in the shallow waters. However, in this chapter a general analysis similar to those in W. M. Charles et al. (2009) shows that a process observed over a long time spans as modelled by the coloured noise force behaves much like a Brownian motion model with variance parameter $\sigma^2 T_L^2 \prod_{i=1}^{n} \alpha_i^2$. The use of coloured noise however is more expensive in terms of computation and therefore it is advisable to use the particle model driven by coloured for short term behaviour while adhering to the traditional particle model for long-term simulations.

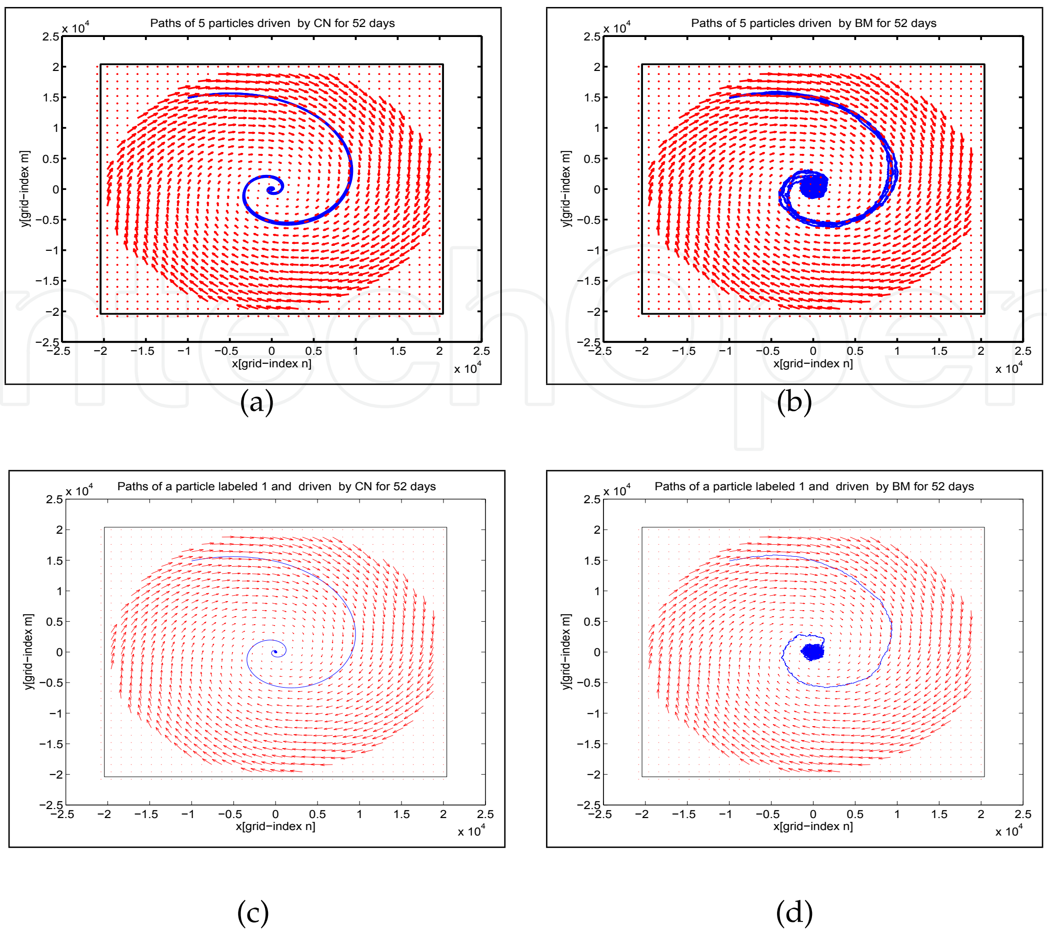


Fig. 5. Tracking of a single marked particle in the whirl pool starting from the location

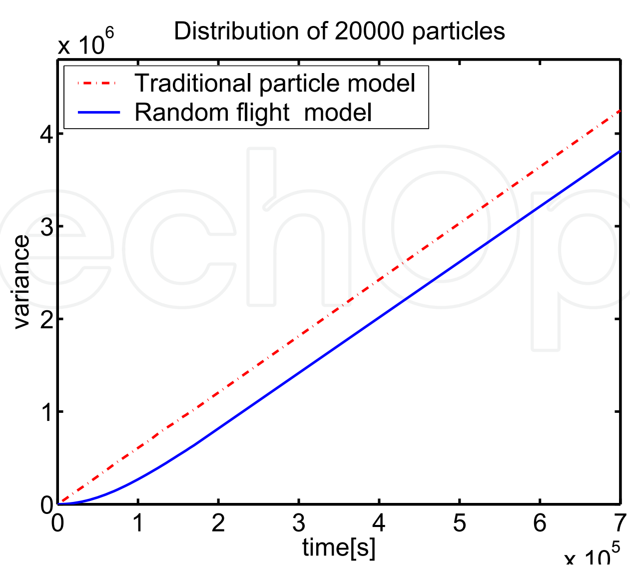


Fig. 6. The variance of a cloud of 20,000 particles in the idealized empty domain. There are two zones, one in which the variance grows quadratically with time for $t << T_L$ and another one it grows linearly with time for $t >> T_L$.

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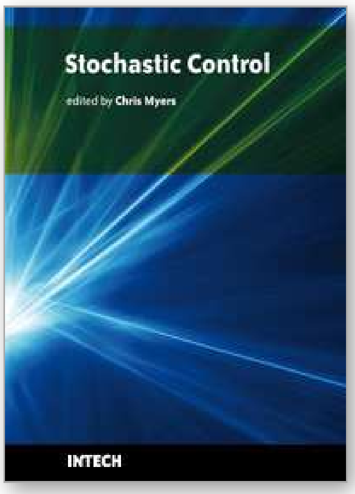

Stochastic Control

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