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The Itô calculus for a noisy dynamical system

Shambhu N. Sharma
Department of Electrical Engineering
National Institute of Technology, Surat, India
snsvolterra@gmail.com

The deterministic versions of dynamical systems have been studied extensively in literature. The notion of noisy dynamical systems is attributed to random initial conditions and small perturbations felt by dynamical systems. The stochastic differential equation formalism is utilized to describe noisy dynamical systems. The Itô calculus, a pioneering contribution of Kiyoshi Itô, is regarded as a path-breaking discovery in the branch of mathematical science in which the term $'dB_t' = \dot{B}_t dt$, where the Brownian motion $B = \{B_t, t_0 \le t < \infty\}$. The Itô theory deals with multi-dimensional Itô differential rule, Itô stochastic integral and subsequently, can be exploited to analyse non-linear stochastic differential systems.

This chapter discusses the usefulness of Itô theory to analysing a noisy dynamical system. In this chapter, we consider a system of two coupled second-order fluctuation equations, which has central importance in noisy dynamical systems. Consider the system of the coupled fluctuation equations of the form

$$\ddot{x}_1 = F_1(t, x_1, \dot{x}_1, x_2, \dot{x}_2, \dot{B}_1),$$
$$\ddot{x}_2 = F_2(t, x_1, \dot{x}_1, x_2, \dot{x}_2, \dot{B}_2),$$

where the state vector $x_t = (x_1, x_2, \dot{x}_1, \dot{x}_2)^T$ and the vector Brownian motion $B_t = (B_1, B_2)^T$. Interestingly, a suitable choice of the right-hand side terms F_1, F_2 of the above formalism describes the motion of an orbiting satellite in noisy environment, which w'd be the subject of discussion. After accomplishing the phase space formulation, the structure of the dynamical system of concern here becomes a multi-dimensional stochastic differential equation. Remarkably, in this chapter, the resulting SDE is analysed using the Itô differential rule in contrast to the Fokker-Planck approach. This chapter aims to open the topic to a broader audience as well as provides guidance for understanding the estimation-theoretic scenarios of stochastic differential systems.

Key words: Brownian motion, Itô differential rule, Fokker-Planck approach, second-order fluctuation equations, multi-dimensional stochastic differential equation

1. Introduction

The Ordinary Differential Equation (ODE) formalism is utilized to analyse dynamical systems deterministically. After accounting the effect of random initial conditions and small perturbations felt by dynamical systems gives rise to the concept of stochastic processes and subsequently, stochastic differential equations, a branch of mathematical science. As a result of these, the SDE confirms actual physical situations in contrast to the ODE. A remarkable success of stochastic differential equations can be found in different branches of sciences, i.e. stochastic control, satellite trajectory estimations, helicopter rotor, stochastic networks, mathematical finance, blood clotting dynamics, protein kinematics, population dynamics, neuronal activity. A nice exposition about the application of stochastic processes and Stochastic Differential Equations in sciences can be found in celebrated books authored by Karatzas and Shreve (1991), Kloeden and Platen (1991), Campen (2007) . The stochastic differential equation in the Itô sense is a standard form to describe dynamical systems in noisy environments. Alternatively, stochastic differential equations can be re-written involving $\frac{1}{2}$ differential, i.e. the Stratonovich sense, as well as p-differential, where $0 \leq p \leq 1$ (Pugachev and Synstin 1977). The Itô stochastic differential equation describes stochastic differential systems driven by the Brownian motion process. The Brownian motion process has greater conceptual depth and ageless beauty. The Brownian motion process is a Gauss-Markov process as well as satisfies the martingale properties, i.e. $E(x_t|F_s) = x_s, t \geq s$ and the sigma algebra $F_s = \bigcup_{r \leq s} F_r$ (Revuz and Yor 1991, Strook and Varadhan 1979). The Central Limit Theorem (CLT) of stochastic processes confirms the usefulness of the Brownian motion for analysing randomly perturbed dynamical systems. The Brownian motion process can be utilized to generate the Ornstein-Uhlenbeck (OU) process, a colored noise (Wax 1954). This suggests that the stochastic differential system driven by the OU process can be reformulated as the Itô stochastic differential equation by introducing the notion of 'augmented state vector approach'. Moreover, the state vector, which satisfies the stochastic differential equation driven by the OU process, will be non-Markovian. On the other hand, the augmented state vector, after writing down the SDE for the OU process, becomes the Markovian. For these reasons, the Itô stochastic differential equation would be the cornerstone formalism in this chapter. The white noise can be regarded as informal non-existent time derivative $\dot{B}_t$ of the Brownian motion B_t. Kiyoshi Ito considered the term 'dB_t' resulting from the multiplication between the white noise $\dot{B}_t$ and the time differential dt.

This chapter demonstrates the usefulness of the Itô theory to analysing the motion of an orbiting satellite accounting for stochastic accelerations. Without accounting the effect of stochastic accelerations, stochastic estimation algorithms may lead to inaccurate estimation of positioning of the orbiting particle.

After introducing the phase space formulation, the stochastic problem of concern here can be regarded as a dynamical system perturbed by the Brownian motion process. In this chapter, the multi-dimensional Itô differential rule is exploited to analyse the stochastic differential system, which is the subject of discussion, in contrast to the Fokker-Planck

approach (Sharma and Parthasarathy 2007). The Fokker-Planck Equation (FPE) is a parabolic linear homogeneous differential equation of order two in partial differentiation for the transition probability density. A discussion on the Fokker-Planck equation is given in appendix 2. The chapter encompasses estimation-theoretic scenarios as well as qualitative analysis of the stochastic problem considered here.
This chapter is organized as follows: section (2) begins by writing a generalized structure of two-coupled second-order fluctuation equations. Subsequently, approximate evolutions of conditional mean vector and variance matrix are derived. In section (3), numerical experiments were accomplished. Concluding remarks are given in section (4). Furthermore, a qualitative analysis of the stochastic problem of concern here can be found in 'appendix'1.

2. The structure of a noisy dynamical system and evolution equations

In dynamical systems' theory, second-order fluctuation equations describe dynamical systems perturbed by noise processes. Here, first we consider a system of two coupled second-order equations, which is an appealing case in dynamical systems and the theory of ordinary differential equations (Arnold 1995),

$$\ddot{x}_1 = F_1(t, x_1, \dot{x}_1, x_2, \dot{x}_2),$$
$$\ddot{x}_2 = F_2(t, x_1, \dot{x}_1, x_2, \dot{x}_2),$$

after introducing the noise processes along the components (x_1, x_2) of the coupled equations, the above can be re-written as

$$\ddot{x}_1 = F_1(t, x_1, \dot{x}_1, x_2, \dot{x}_2, \dot{B}_1), \tag{1}$$
$$\ddot{x}_2 = F_2(t, x_1, \dot{x}_1, x_2, \dot{x}_2, \dot{B}_2). \tag{2}$$

Equations (1)-(2) constitute a system of two coupled second-order fluctuation equations. After accomplishing the phase space formulation, the above system of fluctuation equations leads to a multi-dimensional stochastic differential equation. Choose

$$\dot{x}_1 = x_3,$$
$$\dot{x}_2 = x_4,$$

and

$$\dot{x}_3 = F_1(t, x_1, x_2, x_3, x_4, \dot{B}_1),$$
$$\dot{x}_4 = F_2(t, x_1, x_2, x_3, x_4, \dot{B}_2).$$

By considering a special case of the above system of equations, we have

$$dx_1 = x_3 dt,$$
$$dx_2 = x_4 dt,$$

and

$$dx_3 = f_3(t, x_1, x_2, x_3, x_4)dt + g_3(t, x_1, x_2, x_3, x_4)dB_1,$$
$$dx_3 = f_4(t, x_1, x_2, x_3, x_4)dt + g_4(t, x_1, x_2, x_3, x_4)dB_2.$$

The resulting stochastic differential equation is a direct consequence of the Itô theory, i.e. $'dB_t' = \dot{B}_t dt$. More precisely,

$$dx_t = f(t, x_1, x_2, x_3, x_4)dt + G(t, x_1, x_2, x_3, x_4)dB_t, \tag{3}$$

where

$$x_t = (x_1, x_2, x_3, x_4)^T,\ f(t, x_1, x_2, x_3, x_4) = (x_3,\ x_4, f_3, f_4)^T,$$

$$G(t, x_1, x_2, x_3, x_4) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ g_3 & 0 \\ 0 & g_4 \end{pmatrix},\ dB_t = (dB_1, dB_2)^T.$$

Equation (3) can be regarded as the stochastic differential equation in the Itô sense. Alternatively, the above stochastic differential equation can be expressed in the Stratonovich sense. The Stratonovich stochastic differential equation can be re-written as the Itô stochastic differential equation using mean square convergence. A greater detail can be found in Jazwinski (1970), Protter (2005) and Pugachev and Synstin (1977). Here, the Itô SDE w'd be the cornerstone formalism for the stochastic problem of concern here. It is interesting to note that the motion of an orbiting particle accounting for stochastic dust particles' perturbations can be modeled in the form of stochastic differential equation, i.e. equation (3), where

$$x_t = (x_1, x_2, x_3, x_4)^T = (r,\ \phi,\ v_r,\ \omega)^T,$$
$$f(x_t, t) = (x_3, x_4, f_3, f_4)^T = (v_r,\ \omega,\ (\omega^2 r - V'(r)),\ -\frac{2v_r\omega}{r})^T,$$
$$G(x_t, t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \sigma_r r & 0 \\ 0 & \dfrac{\sigma_\phi}{r} \end{pmatrix}, \tag{4}$$

and r, ϕ are the radial and angular co-ordinates respectively. The radial and angular components of the stochastic velocity are $\sigma_r r\, dB_r$ and $\frac{\sigma_\phi}{r} dB_\phi$ respectively. A procedure for deriving the equation of motion of the stochastic differential system of concern here involves the following: (i) write down the Lagrangian of the orbiting particle

$$L(r, \dot{r}, \dot{\phi}) = \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\phi}^2) - mV(r).$$

This form of the Lagrangian is stated in Landau (1976), which results from the Lagrangian $L(r, \theta, \dot{r}, \dot{\theta}, \dot{\phi}) = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2(\sin\theta)^2 \dot{\phi}^2) - mV(r)$ evaluated at $\theta = \frac{\pi}{2}$.
(ii) Subsequently, the use of the Euler-Lagrange equation with additional random forces along $(r(t), \phi(t))$ results stochastic two-body dynamics, a system of two coupled second-order fluctuation equations assuming the structure of equations (1)-(2) (iii) accomplish phase space formulation, which leads to the multi-dimentional stochastic differential equation. For a greater detail about the motion of the orbiting particle in a stochastic dust environment, the Royal Society paper (Sharma and Partasarathy 2007) can be consulted. A theoretical justification explaining 'why the Brownian motion process is accurate to describe the dust perturbation' hinges on the Central Limit Theorem of stochastic processes.
Equation (3) in conjunction with equation (4) can be re-stated in the standard format as

$$dx_t = f(x_t, t)dt + G(x_t, t, \sigma_r, \sigma_\phi)dB_t,$$

where x_t is the state vector, $f(x_t, t)$ is the system non-linearity, $G(x_t, t, \sigma_r, \sigma_\phi)$ is the dispersion matrix, σ_r and σ_ϕ are diffusion parameters. The on-line estimation of the diffusion parameters σ_r and σ_ϕ can be accomplished from experiments by taking measurements on the particle trajectory at discrete-time instants using the Maximum Likelihood Estimate (MLE). The MLE involves the notion of the conditional probability density $p(z_{\tau_n}, z_{\tau_{n-1}}, \ldots z_{\tau_2}, z_{\tau_1}, z_{\tau_0} | \sigma_r, \sigma_\phi)$, where z_{τ_i} denotes the observation vector at i th time instant, $0 \le i \le n$. The estimated parameter vector $(\sigma_r, \sigma_\phi)^T = \max_{\sigma_r, \sigma_\phi} p(z_{\tau_n}, z_{\tau_{n-1}}, \ldots z_{\tau_2}, z_{\tau_1}, z_{\tau_0} | \sigma_r, \sigma_\phi)$. Moreover, the conditional probability density $p(z_{\tau_n}, z_{\tau_{n-1}}, \ldots z_{\tau_2}, z_{\tau_1}, z_{\tau_0} | \sigma_r, \sigma_\phi)$ can be regarded as the conditional expectation of the conditional probability density $p(z_{\tau_n}, z_{\tau_{n-1}}, \ldots z_{\tau_2}, z_{\tau_1}, z_{\tau_0} | x_{\tau_n}, x_{\tau_{n-1}}, x_{\tau_{n-2}}, \ldots x_{\tau_2}, x_{\tau_1}, x_{\tau_0})$, i.e.

$$p(z_{\tau_n}, z_{\tau_{n-1}}, \ldots, z_{\tau_2}, z_{\tau_1}, z_{\tau_0} \big| \sigma_r, \sigma_\phi)$$
$$= E(p(z_{\tau_n}, z_{\tau_{n-1}}, \ldots, z_{\tau_2}, z_{\tau_1}, z_{\tau_0} \big| x_{\tau_n}, x_{\tau_{n-1}}, x_{\tau_{n-2}}, \ldots x_{\tau_2}, x_{\tau_1}, x_{\tau_0}) \big| \sigma_r, \sigma_\phi),$$

where

$$x_{\tau_{k+1}} = x_{\tau_k} + f(x_{\tau_k}, \tau_k)(\tau_{k+1} - \tau_k) + \sqrt{\tau_{k+1} - \tau_k}\; w_{\tau_{k+1}},\ w_{\tau_{k+1}} \text{ is } N(0,1).$$

After determining the diffusion parameters on the basis of the MLE, the diffusion parameters are plugged into the above diffusion equation, i.e. stochastic differential equation. As a result of this, we have

$$dx_t = f(x_t, t)dt + \mathrm{G}(x_t, t)\, dB_t, \tag{5}$$

where $dB_t \sim N(0, Idt)$. A detailed discussion about the on-line estimation of unknown parameters of the stochastic differential system can be found in Dacunha-Castelle and Florens-Zmirou (1986). The above stochastic differential equation, equation (5), in conjunction with equation (4) can be analysed using the Fokker-Planck approach. Making the use of the FPE, we derive the evolution of the conditional moment, conditional expectation of the scalar function of an n-dimensional state vector. Note that the Fokker-Planck operator is an adjoint operator.

This chapter is intended to analyse the stochastic problem of concern here using the multi-dimensional Itô differential rule in contrast to the FPE approach. Here, we explain the Itô theory briefly and subsequently, its usefulness for analysing the noisy dynamical system. Consider the state vector $x_t = (x_1, x_2)^T \in U$ is a solution vector of the above SDE, $\phi : U \to R$, i.e. $\phi(x_t) \in R$, and the phase space $U \subset R^n$. Suppose the function $\phi(x_t)$ is twice differentiable. The stochastic evolution $d\phi(x_t)$ of the scalar function of the n-dimensional state vector using the stochastic differential rule can be stated as

$$d\phi(x_t) = \sum_i \frac{\partial \phi(x_t)}{\partial x_i} dx_i + \frac{1}{2} \sum_{i,j} dx_i dx_j \frac{\partial^2 \phi(x_t)}{\partial x_i \partial x_j}.$$

After plugging the ith component of stochastic differential equation, i.e. equation (5), in the above evolution, we have

$$d\phi(x_t) = (\sum_i \frac{\partial \phi(x_t)}{\partial x_i} f_i(x_t, t) + \frac{1}{2} \sum_i (GG^T)_{ii}(x_t, t) \frac{\partial^2 \phi(x_t)}{\partial x_i^2}$$

$$+\sum_{i<j}(GG^T)_{ij}(x_t,t)\frac{\partial^2\phi(x_t)}{\partial x_i\partial x_j})dt \quad +\sum_{1\le i\le n,1\le\gamma\le r}\frac{\partial\phi(x_t)}{\partial x_i}G_{i\gamma}(x_t,t)dB_\gamma, \qquad (6)$$

where the size of the vector Brownian motion process is r. Note that the contribution to the term $d\phi(x_t)$ coming from the second and third terms of the right-hand side of equation (6) is attributed to the property $dB_\phi dB_\gamma = \delta_{\phi\gamma}dt$. The integral counterpart of equation (6) can be written as

$$\phi(x_t)=\phi(x_{t_0})+(\sum_i\int_{t_0}^{t}\frac{\partial\phi(x_s)}{\partial x_i(s)}f_i(x_s,s)ds+\frac{1}{2}\sum_i\int_{t_0}^{t}(GG^T)_{ii}(x_s,s)\frac{\partial^2\phi(x_s)}{\partial x_i^2(s)}ds$$

$$+\sum_{i<j}\int_{t_0}^{t}(GG^T)_{ij}(x_s,s)\frac{\partial^2\phi(x_s)}{\partial x_i(s)\partial x_j(s)}ds)$$

$$+\sum_{1\le i\le n,1\le\gamma\le r}\int_{t_0}^{t}\frac{\partial\phi(x_s)}{\partial x_i}G_{i\gamma}(x_s,s)dB_\gamma(s).$$

The evolution $d\widehat{\phi}(x_t)$ of the conditional moment is the standard formalism to analyse stochastic differential systems. The contribution to the term $d\widehat{\phi}(x_t)$ comes from the system non-linearity and dispersion matrix, since the term $\phi(x_t)$ is a scalar function of the $n-$dimensional state vector. The state vector satisfies the Itô stochastic differential equation, see equation (5). As a result of this, the expectation and differential operators can be interchanged.

$$d\widehat{\phi}(x_t)=E(d\,\phi(x_t)|x_{t_0},t_0).$$

The above equation in conjunction with equation (6) leads to

$$d\widehat{\phi}(x_t)=(\sum_p\overbrace{f_p(x_t,t)\frac{\partial\phi(x_t)}{\partial x_p}}^{\wedge}+\frac{1}{2}\sum_p\overbrace{(GG^T)_{pp}(x_t,t)\frac{\partial^2\phi(x_t)}{\partial x_p^2}}^{\wedge}+\sum_{p<q}\overbrace{(GG^T)_{pq}(x_t,t)\frac{\partial^2\phi(x_t)}{\partial x_p\partial x_q}}^{\wedge})dt.$$

Note that the expected value of the last term of the right-hand side of equation (6) vanishes, i.e. $\langle G_{i\gamma}(x_t,t)dB_\gamma\rangle=0.$ For the exact mean and variance evolutions, we consider $\phi(x_t)$ as $x_i(t)$ and $\tilde{x}_i\tilde{x}_j$ respectively, where $\tilde{x}_i=x_i-\hat{x}_i$. Thus, we have

$$d\widehat{x}_i(t) = \widehat{f}_i(x_t, t)dt, \tag{7}$$

$$dP_{ij} = (\overbrace{x_i f_j}^{\wedge} - \widehat{x}_i \widehat{f}_j + \overbrace{f_i x_j}^{\wedge} - \widehat{f}_i \widehat{x}_j + \overbrace{(GG^T)_{ij}(x_t, t)}^{\wedge})dt, \tag{8}$$

where

$$\begin{aligned}\widehat{x}_i(t) &= E(x_i(t)\big|x_{t_0}, t_0),\\ P_{ij} &= E((x_i - E(x_i(t)\big|x_{t_0}, t_0))(x_j - E(x_j\big|x_{t_0}, t_0))\big|x_{t_0}, t_0)\\ &= E(\widetilde{x}_i\widetilde{x}_j\big|x_{t_0}, t_0).\end{aligned}$$

The analytical and numerical solutions of the exact estimation procedure for the non-linear stochastic differential system are not possible, since its evolutions are infinite dimensional and require knowledge of higher-order moment evolutions. For these reasons, approximate evolutions, which preserve some of the qualitative characteristics of the exact evolutions, are analysed. Here, the bilinear and second-order approximations are the subject of investigation. The second-order approximate evolution equations can be derived by introducing second-order partials of the system non-linearity $f(x_t, t)$ and the diffusion coefficient $(GG^T)(x_t, t)$ into the exact mean and variance evolutions, equations (7)-(8). Thus, the mean and variance evolutions for the non-linear stochastic differential system, using the second-order approximation, are

$$d\widehat{x}_i = (f_i(\widehat{x}_t, t) + \frac{1}{2}\sum_{p,q} P_{pq} \frac{\partial^2 f_i(\widehat{x}_t, t)}{\partial\widehat{x}_p \partial\widehat{x}_q})dt, \tag{9}$$

$$\begin{aligned}(dP_t)_{ij} = (&\sum_p P_{ip} \frac{\partial f_j(\widehat{x}_t, t)}{\partial \widehat{x}_p} + \sum_p P_{jp} \frac{\partial f_i(\widehat{x}_t, t)}{\partial \widehat{x}_p} + (GG^T)_{ij}(\widehat{x}_t, t)\\ &+ \frac{1}{2}\sum_{p,q} P_{pq} \frac{\partial^2 (GG^T)_{ij}(\widehat{x}_t, t)}{\partial\widehat{x}_p \partial\widehat{x}_q})dt.\end{aligned} \tag{10}$$

Making the use of the above conditional moment evolutions for the system non-linearity and process noise coefficient matrix stated in equation (4), leads to the following mean and variance evolutions for the stochastic differential system considered here:

Mean evolutions

$$d\hat{r} = \hat{v}_r dt\,, \quad d\hat{\phi} = \hat{\omega} dt,$$

$$d\hat{v}_r = (\hat{r}\hat{\omega}^2 - V'(\hat{r}) - \frac{1}{2}\frac{\partial^2 V'(\hat{r}) P_{rr}}{\partial \hat{r}^2} + \hat{r} P_{\omega\omega} + 2\hat{\omega} P_{r\omega}) dt,$$

$$d\hat{\omega} = (-\frac{2\hat{v}_r\hat{\omega}}{\hat{r}} + \frac{2\hat{\omega} P_{rv_r}}{\hat{r}^2} + \frac{2\hat{v}_r P_{r\omega}}{\hat{r}^2} - \frac{2\ \hat{v}_r\ \hat{\omega}\, P_{rr}}{\hat{r}^3} - \frac{2P_{v_r\omega}}{\hat{r}}) dt.$$

Variance evolutions

$$dP_{rr} = 2P_{rv_r} dt, \qquad dP_{r\phi} = (P_{v_r\phi} + P_{r\omega}) dt,$$

$$dP_{rv_r} = ((\hat{\omega}^2 - \frac{\partial V'(\hat{r})}{\partial \hat{r}}) P_{rr} + 2\hat{r}\hat{\omega} P_{r\omega} + P_{v_r v_r}) dt,$$

$$dP_{r\omega} = (\frac{2\hat{r}\hat{\omega} P_{rr}}{\hat{r}^2} - \frac{2\hat{\omega} P_{rv_r}}{\hat{r}} - 2\frac{\hat{v}_r P_{r\omega}}{\hat{r}} + P_{v_r\omega}) dt,$$

$$dP_{\phi\phi} = 2P_{\phi\omega} dt\ \ ,\ dP_{\phi v_r} = ((\hat{\omega}^2 - \frac{\partial V'(\hat{r})}{\partial \hat{r}}) P_{r\phi} + 2\hat{r}\,\hat{\omega} P_{\phi\omega} + P_{v_r\omega}) dt,$$

$$dP_{\phi\omega} = (\frac{2\hat{v}_r\hat{\omega} P_{r\phi}}{\hat{r}^2} - \frac{2\hat{\omega} P_{\phi v_r}}{\hat{r}} - 2\frac{\hat{v}_r P_{\phi\omega}}{\hat{r}} + P_{\omega\omega}) dt,$$

$$dP_{v_r v_r} = (2(\hat{\omega}^2 - \frac{\partial V'(\hat{r})}{\partial \hat{r}}) P_{rv_r} + 4\hat{r}\hat{\omega} P_{v_r\omega} + \sigma_r^2 \hat{r}^2 + \sigma_r^2 P_{rr}) dt,$$

$$dP_{\omega v_r} = (\frac{2\hat{v}_r\,\hat{\omega} P_{rv_r}}{\hat{r}^2} - \frac{2\hat{\omega} P_{v_r v_r}}{\hat{r}} - 2\frac{\hat{v}_r P_{v_r\omega}}{\hat{r}} + (\hat{\omega}^2 - \frac{\partial V'(\hat{r})}{\partial \hat{r}}) P_{r\omega} + 2\hat{r}\hat{\omega} P_{\omega\omega}) dt$$

$$dP_{\omega\omega} = (\frac{4\hat{v}_r\,\hat{\omega} P_{r\omega}}{\hat{r}^2} - 4\frac{\hat{\omega} P_{v_r\omega}}{\hat{r}} - 4\frac{\hat{v}_r P_{\omega\omega}}{\hat{r}} + \frac{\sigma_\phi^2}{\hat{r}^2} + 3\sigma_\phi^2 \frac{P_{rr}}{\hat{r}^4})\,.$$

The second-order approximation of the function f, and the diffusion coefficient matrix

$GG^T(x_t,t)$ around the mean trajectory gives

$$f(x_t,t) \approx f(\hat{x}_t,t) + \sum_p \tilde{x}_p \frac{\partial f(\hat{x}_t,t)}{\partial \hat{x}_p} + \frac{1}{2}\sum_{p,q} \tilde{x}_p \tilde{x}_q \frac{\partial^2 f(\hat{x}_t,t)}{\partial \hat{x}_p \partial \hat{x}_q},$$

$$GG^T(x_t,t) \approx GG^T(\hat{x}_t,t) + \sum_p \tilde{x}_p \frac{\partial GG^T(\hat{x}_t,t)}{\partial \hat{x}_p} + \frac{1}{2}\sum_{p,q} \tilde{x}_p \tilde{x}_q \frac{\partial^2 GG^T(\hat{x}_t,t)}{\partial \hat{x}_p \partial \hat{x}_q},$$

where $\tilde{x}_p = x_p - \hat{x}_p$. The approximate conditional moment evolutions using bilinear approximation can be obtained by considering only the first-order partials of the system non-linearity and diffusion coefficients. In other words, the terms $\frac{1}{2}\sum_{p,q}\tilde{x}_p\tilde{x}_q \frac{\partial^2 f(\hat{x}_t,t)}{\partial \hat{x}_p \partial \hat{x}_q}$ and $\frac{1}{2}\sum_{p,q}\tilde{x}_p\tilde{x}_q \frac{\partial^2 GG^T(\hat{x}_t,t)}{\partial \hat{x}_p \partial \hat{x}_q}$ vanish for the bilinear approximation. As a result of this,

$$d\hat{r} = \hat{v}_r dt, \quad d\hat{\phi} = \hat{\omega} dt,$$

$$d\hat{v}_r = (\hat{r}\hat{\omega}^2 - V'(\hat{r}))dt, \quad d\hat{\omega} = (-\frac{2\hat{v}_r\hat{\omega}}{\hat{r}})dt,$$

and

$$dP_{rr} = 2P_{rv_r}dt, \qquad dP_{r\phi} = (P_{v_r\phi} + P_{r\omega})dt,$$

$$dP_{rv_r} = ((\hat{\omega}^2 - \frac{\partial V'(\hat{r})}{\partial \hat{r}})P_{rr} + 2\hat{r}\hat{\omega}P_{r\omega} + P_{v_r v_r})dt,$$

$$dP_{r\omega} = (\frac{2\hat{r}\hat{\omega}P_{rr}}{\hat{r}^2} - \frac{2\hat{\omega}P_{rv_r}}{\hat{r}} - 2\frac{\hat{v}_r P_{r\omega}}{\hat{r}} + P_{v_r\omega})dt,$$

$$dP_{\phi\phi} = 2P_{\phi\omega}dt, \quad dP_{\phi v_r} = ((\hat{\omega}^2 - \frac{\partial V'(\hat{r})}{\partial \hat{r}})P_{r\phi} + 2\hat{r}\hat{\omega}P_{\phi\omega} + P_{v_r\omega})dt,$$

$$dP_{\phi\omega} = (\frac{2\hat{v}_r\hat{\omega}P_{r\phi}}{\hat{r}^2} - \frac{2\hat{\omega}P_{\phi v_r}}{\hat{r}} - 2\frac{\hat{v}_r P_{\phi\omega}}{\hat{r}} + P_{\omega\omega})dt,$$

$$dP_{v_r v_r} = (2(\hat{\omega}^2 - \frac{\partial V'(\hat{r})}{\partial \hat{r}})P_{rv_r} + 4\hat{r}\hat{\omega}P_{v_r\omega} + \sigma_r{}^2\hat{r}^2)dt,$$

$$dP_{\omega v_r} = (\frac{2\hat{v}_r\hat{\omega}P_{rv_r}}{\hat{r}^2} - \frac{2\hat{\omega}P_{v_r v_r}}{\hat{r}} - 2\frac{\hat{v}_r P_{v_r\omega}}{\hat{r}} + (\hat{\omega}^2 - \frac{\partial V'(\hat{r})}{\partial \hat{r}})P_{r\omega} + 2\hat{r}\hat{\omega}P_{\omega\omega})dt,$$

$$dP_{\omega\omega} = (\frac{4\hat{v}_r\hat{\omega}P_{r\omega}}{\hat{r}^2} - 4\frac{\hat{\omega}P_{v_r\omega}}{\hat{r}} - 4\frac{\hat{v}_r P_{\omega\omega}}{\hat{r}} + \frac{\sigma_\phi{}^2}{\hat{r}^2})dt.$$

The mean trajectory for the stochastically perturbed dynamical system using bilinear approximation does not include *variance terms* in the mean evolution. The term $\left\langle GG^T(x(t),t)\right\rangle$, the expected value of the diffusion coefficient, in the variance evolution accounts for the stochastic perturbation felt by the orbiting particle. For this reason, the bilinear approximation leads to the 'unperturbed mean trajectory', see figures (1)-(4) as well. On the other hand, the variance evolution using bilinear approximation for the dust-perturbed model includes perturbation effects, i.e. $GG^T(\hat{x}_t,t)$. In order to account for the stochastic perturbation in the mean evolution, we utilize the second-order approximation in the mean evolution. The second-order approximation includes 'the second-order partials' of the system non-linearity $f(x_t,t)$ and variance terms in the mean trajectory, which leads to better *estimation* of the trajectory. The variance evolution dP_{v_r} of the radial velocity, using the second-order approximation, involves an additional term $\sigma_r{}^2 P_{rr}$ in contrast to the bilinear approximation. The variance evolution dP_ω of the angular velocity, using the second-order approximation, accounts for a correction term $3\sigma_\phi{}^2\frac{P_{rr}}{\hat{r}^4}$, in contrast to the bilinear approximation as well.

Note that the conditional moment evolutions derived in this chapter for the stochastic problem of concerns here agree with the evolutions stated in a Royal society paper (Sharma and Parthasarathy 2007). However, the approach of this chapter, multi-dimensional Itô rule, is different from the Fokker-Plank approach adopted in the Royal Society contribution.

3. Numerical experiments

The simulations of the mean and variance evolutions are accomplished using a simple, but effective finite difference method-based numerical scheme. The discrete version of the standard stochastic differential equation is

$$x_{t_{k+1}} = x_{t_k} + f(x_{t_k}, t_{\mathrm{k}})(t_{k+1} - t_k) + \mathrm{G}(x_{t_k}, t_{\mathrm{k}})\sqrt{t_{k+1} - t_k}\,\mathrm{w}_{\mathrm{t}_{\mathrm{k}+1}},$$

where $w_{t_{k+1}}$ is a standard normal variable. The dimension of the phase space of the stochastic problem of concern here is four, since the state vector $x_t = (x_1, x_2, x_3, x_4)^T = (r, \phi, v_r, \omega)^T \in U \subset R^4$. The size of the mean state vector is four and the number of entries in the variance matrix of the state is sixteen. Since the state vector is a real-valued vector stochastic process, the condition $P_{ij} = P_{ji}$ holds. The total number of distinct entries in the variance matrix w'd be ten. The initial conditions are chosen as

$$\hat{r}(0) = 1\,\mathrm{AU}, \hat{\phi}(0) = 1\,\mathrm{rad},\ \hat{v}_r(0) = 0.01\,\mathrm{AU/TU}, \hat{\omega}(0) = 1.1\ \mathrm{rad/TU}, \mathrm{P}_{xy}(0) = 0,$$
$$\text{for the state variable } x \text{ and } y.$$

The initial conditions considered here are in canonical system of units. Astronomers adopt a normalized system of units, i.e. 'canonical units', for the simplification purposes. In canonical units, the physical quantities are expressed in terms of Time Unit (TU) and Astronomical Unit (AU). The diffusion parameters $\sigma_r = 0.0121(\mathrm{TU})^{-\frac{3}{2}}$ and $\sigma_\phi = 2.2 \times 10^{-4} \dfrac{AU}{(TU)^{\frac{3}{2}}}$ are chosen for numerical simulations. Here we consider a set of deterministic initial conditions, which implies that the initial variance matrix w'd be zero. Note that random initial conditions lead to the non-zero initial variance matrix. The system is deterministic at $t = t_0$ and becomes stochastic at $t > t_0$ because of the stochastic perturbation. This makes the contribution to the variance evolution coming from the 'system non-linearity coupled with 'initial variance terms' will be zero at $t = t_1$. The contribution to the variance evolution at $t = t_1$ comes from the perturbation term $(GG^T)(x_t, t)$ only. For $t > t_1$, the contribution to the variance evolution comes from the system non-linearity as well as the perturbation term. This assumption allows to study the effect of random perturbations explicitly on the dynamical system. The values of diffusion parameters are selected so that the contribution to the force coming from the random part is smaller than the force coming from the deterministic part. It has been chosen for simulational convenience only.

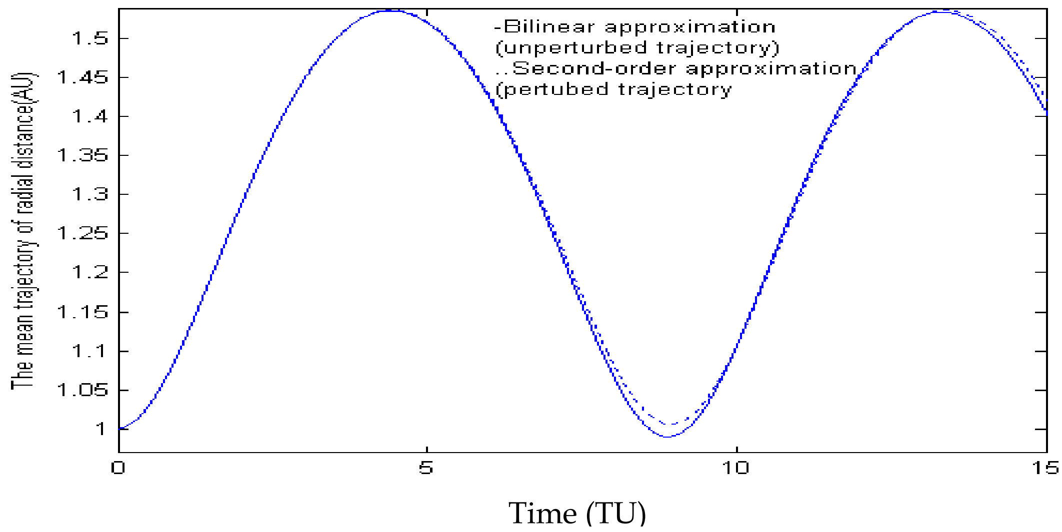


Fig. 1.

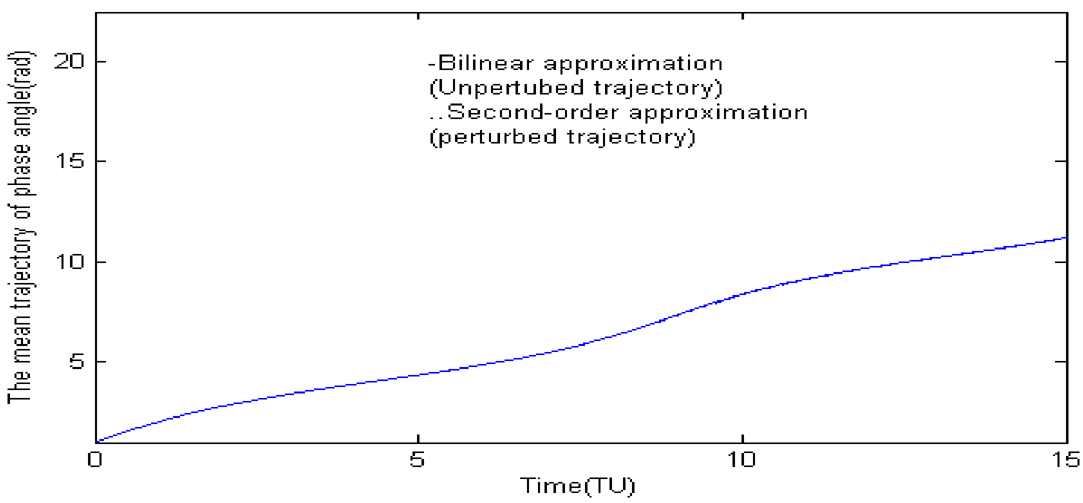


Fig. 2.

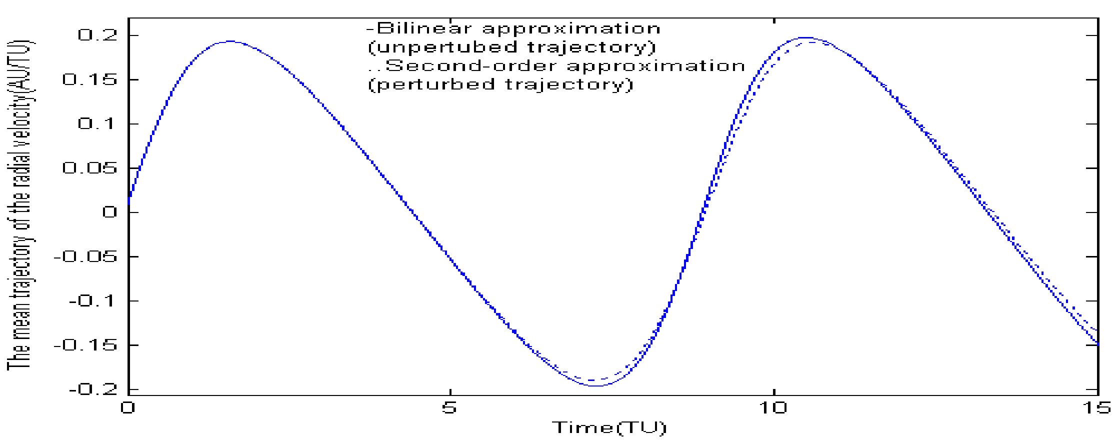
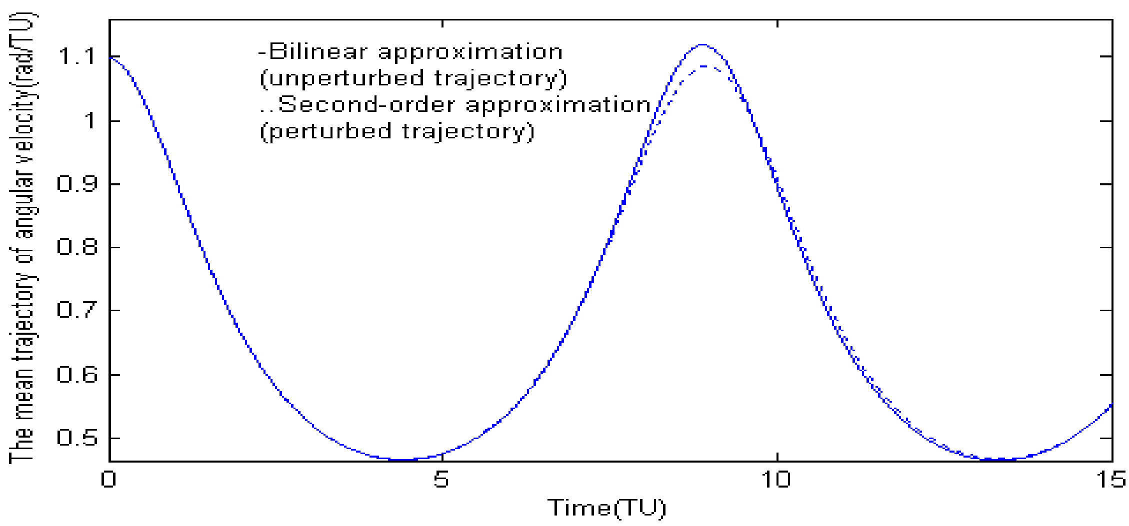


Fig. 3.



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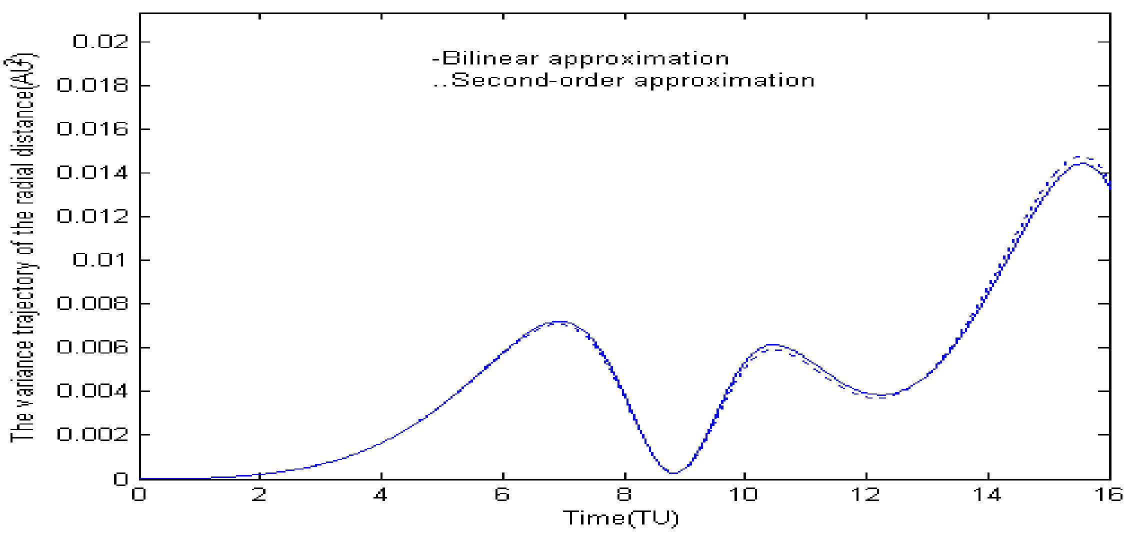

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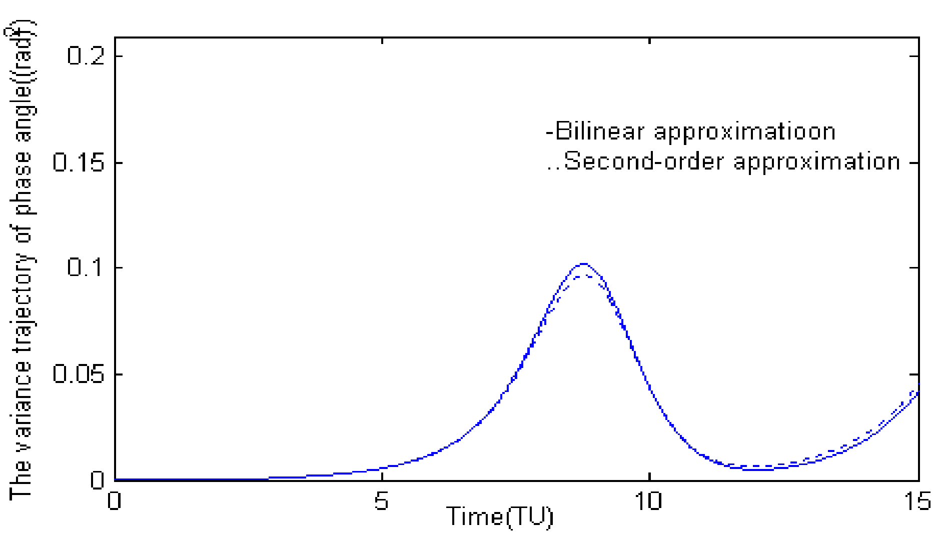

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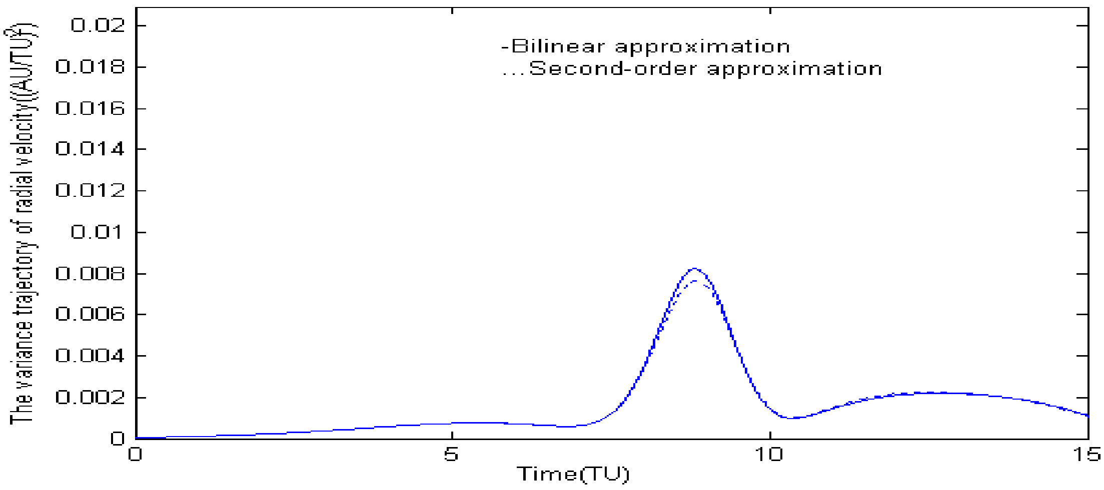

Fig. 7.

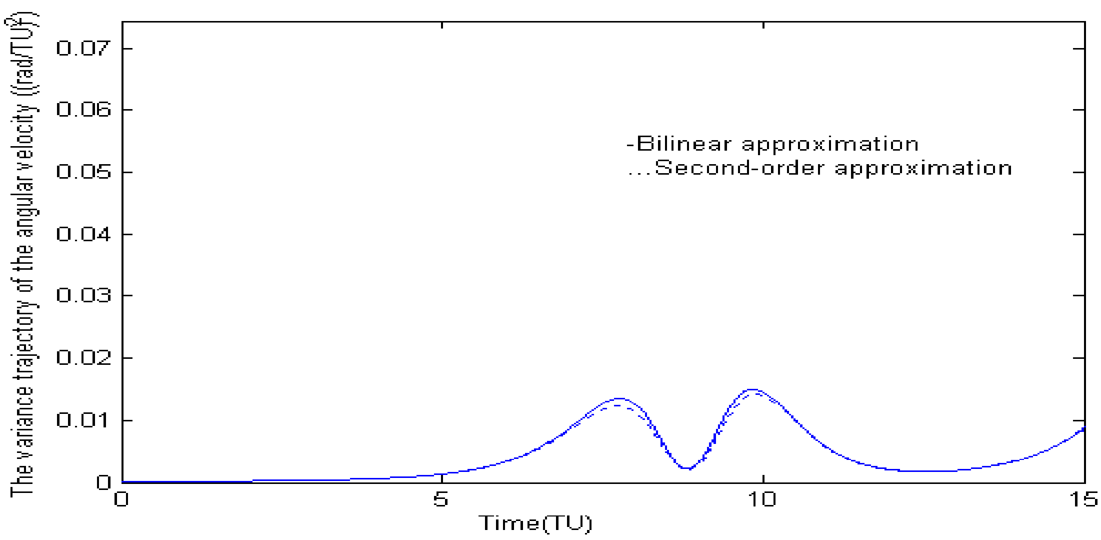

Fig. 8.

Here, we analyse the stochastic problem involving the numerical simulation of approximate conditional moment evolutions. The approximate conditional moment evolutions, i.e. conditional mean and variance evolutions, were derived in the previous section using the second-order and bilinear approximations. The variance evolutions using the second-order approximation result reduced variances of the state variables rather than the bilinear, see figures (5), (6), (7), and (8). These illustrate that the second-order approximation of the mean evolution produces less random fluctuations in the mean trajectory, which are attributed to the second-order partials of the system non-linearity $f(x_t, t)$, i.e. $\frac{1}{2}\sum_{p,q}\tilde{x}_p\tilde{x}_q \frac{\partial^2 f(\hat{x}_t, t)}{\partial \hat{x}_p \partial \hat{x}_q}$. The expectation of the partial terms leads to

$\frac{1}{2}\sum_{p,q} P_{pq} \frac{\partial^2 f(\hat{x}_t,t)}{\partial \hat{x}_p \partial \hat{x}_q}$. The correction term $\frac{1}{2}\sum_{p,q} P_{pq} \frac{\partial^2 f(\hat{x}_t,t)}{\partial \hat{x}_p \partial \hat{x}_q}$ involves the variance term P_{pq}. The evolution of the variance term P_{ij} encompasses the contributions from the preceding variances, partials of the system non-linearity, the diffusion coefficient $(GG^T)_{ij}(\hat{x}_t,t)$ as well as the second-order partial term $\frac{1}{2}\sum_{p,q} P_{pq}(GG^T)_{ij}(\hat{x}_t,t)$. Significantly, the variance terms are also accounted for in the mean trajectory. This explains the second-order approximation leads to the perturbed mean trajectory. This section discusses very briefly about the numerical testing for the mean and variance evolutions derived in the previous section. A greater detail is given in the Author's Royal Society contribution. This chapter is intended to demonstrate the usefulness of the Itô theory for stochastic problems in dynamical systems by taking up an appealing case in satellite mechanics.

4. Conclusion

In this chapter, the Author has derived the conditional moment evolutions for the motion of an orbiting satellite in dust environment, i.e. a noisy dynamical system. The noisy dynamical system was modeled in the form of multi-dimensional stochastic differential equation. Subsequently, the Itô calculus for 'the Brownian motion process as well as the dynamical system driven by the Brownian motion' was utilized to study the stochastic problem of concern here. Furthermore, the Itô theory was utilized to analyse the resulting stochastic differential equation qualitatively. The Markovian stochastic differential system can be analysed using the Kolmogorov-Fokker-Planck Equation (KFPE) as well. The KFPE-based analysis involves the definition of conditional expectation, the adjoint property of the Fokker-Planck operator as well as integration by part formula. On the other hand, the Itô differential rule involves relatively fewer steps, i.e. Taylor series expansion, the Brownian motion differential rule. It is believed that the approach of this chapter will be useful for analysing stochastic problems arising from physics, mathematical finance, mathematical control theory, and technology.

Appendix 1

The qualitative analysis of the non-linear autonomous system can be accomplished by taking the Lie derivative of the scalar function ϕ, where $\phi : U \to R$, U is the phase space of the non-linear autonomous system and $\phi(x_t) \in R.$ The function ϕ is said to be the first integral if the Lie derivative $L_v \phi$ vanishes (Arnold 1995). The problem of analysing the non-linear stochastic differential system qualitatively becomes quite difficult, since it involves multi-dimensional diffusion equation formalism. The Itô differential rule (Liptser and Shirayayev 1977, Sage and Melsa 1971) allows us to obtain the stochastic evolution of the function ϕ. Equation (6) of this chapter can be re-written as

$$\frac{d\phi(x_t)}{dt} = \sum_i \frac{\partial\phi(x_t)}{\partial x_i} f_i(x_t,t) + \frac{1}{2}\sum_i (GG^T)_{ii}(x_t,t)\frac{\partial^2\phi(x_t)}{\partial x_i^2}$$
$$+\sum_{i<j}(GG^T)_{ij}(x_t,t)\frac{\partial^2\phi(x_t)}{\partial x_i \partial x_j} + \sum_{1\le i\le n, 1\le\gamma\le r} \frac{\partial\phi(x_t)}{\partial x_i} G_{i\gamma}(x_t,t)\dot{B}_\gamma .$$

Consider the function $\phi(x_t) = E(x_t)$, where E (.) is the energy function. Thus the stochastic evolution of the energy function (Sharma and Parthasarathy 2007) can be stated as

$$\frac{dE(x_t)}{dt} = \sum_i \frac{\partial E(x_t)}{\partial x_i}\dot{x}_i + \frac{1}{2}\sum_i (GG^T)_{ii}(x_t,t)\frac{\partial^2 E(x_t)}{\partial x_i^2}$$
$$+\sum_{i<j}(GG^T)_{ij}(x_t,t)\frac{\partial^2 E(x_t)}{\partial x_i \partial x_j}.$$

The above evolution for the stochastic differential system of this chapter assumes the following structure:

$$\frac{dE(r,\phi,v_r,\omega)}{dt} = \frac{\partial E(r,\phi,v_r,\omega)}{\partial r}\dot{r} + \frac{\partial E(r,\phi,v_r,\omega)}{\partial \phi}\dot{\phi} + \frac{\partial E(r,\phi,v_r,\omega)}{\partial v_r}\dot{v}_r + \frac{\partial E(r,\phi,v_r,\omega)}{\partial \omega}\dot{\omega}$$
$$+\frac{1}{2}\sum_{i,j}(GG^T)_{ii}(x_t,t)\frac{\partial^2 E(r,\phi,v_r,\omega)}{\partial x_i^2},$$

where $E(r,\phi,v_r,\omega) = \frac{1}{2}(v_r^2 + r^2\omega^2) + V(r)$. A simple calculation will show that

$$\frac{dE(r,\phi,v_r,\omega)}{dt} = (r\omega^2 + V'(r))v_r + (r\omega^2 - V'(r))v_r + \omega r^2(-\frac{2v_r\omega}{r}) + \sigma_r{}^2 r^2 + \sigma_\phi$$
$$= \sigma_r^2 r^2 + \sigma_\phi^2 .$$

Thus the derivative of the energy function for the stochastic system of concern here will not vanish leading to the non-conservative nature of the energy function.

Appendix 2

The Fokker-Planck equation has received attention in literature and found applications for developing the prediction algorithm for the Itô stochastic differential system. Detailed discussions on the Fokker-Planck equation, its approximate solutions and applications in sciences can be found in Risken (1984), Stratonovich (1963). The Fokker-Planck equation is also known as the Kolmogorov forward equation. The Fokker-Planck equation is a special case of the stochastic equation (kinetic equation) as well. The stochastic equation is about the evolution of the conditional probability for given initial states for non-Markov processes. The stochastic equation is an infinite series. Here, we explain how the Fokker-Planck equation becomes a special case of the stochastic equation. The conditional probability density

$$p(x_1, x_2, \ldots\ldots, x_n) = p(x_1 | x_2, x_3, \ldots, x_n) p(x_2 | x_3, x_4, \ldots, x_n) \ldots. p(x_{n-1} | x_n) p(x_n).$$

In the theory of the Markov process, the above can be re-stated as

$$p(x_1, x_2, \ldots\ldots, x_n) = p(x_1 | x_2) p(x_2 | x_3) \ldots. p(x_{n-1} | x_n) p(x_n).$$

Thus,

$$p(x_1, x_2, \ldots\ldots, x_n) = q_{t_1, t_2}(x_1, x_2) q_{t_2, t_3}(x_2, x_3) \ldots. q_{t_{n-1}, t_n}(x_{n-1}, x_n) q(x_n),$$

where $q_{t_{i-1}, t_i}(x_{i-1}, x_i)$ is the transition probability density, $1 \le i \le n$ and $t_{i-1} > t_i$. The transition probability density is the inverse Fourier transform of the conditional characteristic function, i.e.

$$q_{t_{i-1}, t_i}(x_{i-1}, x_i) = \frac{1}{2\pi} \int e^{-iu(x_{i-1}, x_i)} E e^{iu(x_{i-1} - x_i)} du. \tag{11}$$

For deriving the stochastic equation, we consider the conditional probability density $p(x_1 | x_2)$, where

$$p(x_1, x_2) = p(x_1 | x_2) p(x_2).$$

After integrating over the variable x_2, the above equation leads to

$$p(x_1) = \int q_{t_1, t_2}(x_1, x_2) p(x_2) dx_2. \tag{12}$$

Equation (12) in combination with equation (11) leads to

$$p(x_1) = \frac{1}{2\pi} \int e^{-iu(x_1 - x_2)} (E e^{iu(x_1 - x_2)}) p(x_2) dx_2 du. \tag{13}$$

The conditional characteristic function is the conditional moment generating function and the n th order derivative of the conditional characteristic function $Ee^{iu(x_1-x_2)}$ evaluated at the $u=0$ gives the n th order conditional moment. This can be demonstrated by using the definition of the generating function of mathematical science, i.e.

$$\phi(x,u)=\sum_{0\le n}\varphi_n(x)u^n,$$

where $\phi(x,u)$ can be regarded as the generating function of the sequence $\{\varphi_n(x)\}$. As a result of this, the characteristic function $Ee^{iu(x_1-x_2)}=\sum_{0\le n}\frac{(iu)^n}{n!}\left\langle (x_1-x_2)^n\right\rangle$. After introducing the definition of the conditional characteristic function, equation (13) can be recast as

$$p(x_1)=\frac{1}{2\pi}\int e^{-iu(x_1-x_2)}\left(\sum_{0\le n}\frac{(iu)^n}{n!}\left\langle (x_1-x_2)^n\right\rangle\right)p(x_2)dx_2du$$

$$=\sum_{0\le n}\int\frac{1}{n!}\left(\frac{1}{2\pi}\int (iu)^n e^{-iu(x_1-x_2)}du\right)\left\langle (x_1-x_2)^n\right\rangle p(x_2)dx_2. \tag{14}$$

The term $\frac{1}{2\pi}\int e^{-iu(x_1-x_2)}(iu)^n du$ within the second integral sign of equation (14) becomes $(-\frac{\partial}{\partial x_1})^n\delta(x_1-x_2)$ and leads to the probability density

$$p(x_1)=\sum_{0\le n}\int\frac{1}{n!}\left(-\frac{\partial}{\partial x_1}\right)^n\delta(x_1-x_2)\left\langle (x_1-x_2)^n\right\rangle p(x_2)dx_2. \qquad 15)$$

Consider the random variables x_{t_1} and x_{t_2}, where $t_1>t_2$. The time instants t_1 and t_2 can be taken as $t_1=t+\tau, t_2=t$. For the short hand notation, introducing the notion of the stochastic process, taking $x_1=x_\tau, x_2=x$, equation (15) can be recast as

$$p(x_\tau)=\sum_{0\le n}\int\frac{1}{n!}\left(-\frac{\partial}{\partial x_\tau}\right)^n\delta(x_\tau-x)\left\langle (x_\tau-x\ )^n\right\rangle p(x)dx$$

$$=\sum_{0\le n}\int\frac{1}{n!}\left(-\frac{\partial}{\partial x_\tau}\right)^n\delta(x_\tau-x)k_n(x)\tau\, p(x)dx,$$

where $\left\langle \frac{(x_\tau - x)^n}{\tau} \right\rangle = k_n(x)$ and the time interval condition $\tau \to 0$ leads to

$$\underset{\tau \to 0}{Lt} \frac{p(x_\tau) - p(x)}{\tau} = \sum_{1 \le n} \frac{1}{n!} (-\frac{\partial}{\partial x_\tau})^n k_n(x) p(x)$$

or

$$\dot{p}(x) = \sum_{1 \le n} \frac{1}{n!} (-\frac{\partial}{\partial x})^n k_n(x \ ) p(x). \tag{16}$$

The above equation describes the evolution of conditional probability density for given initial states for the non-Markovian process. The Fokker-Plank equation is a stochastic equation with $k_i(x) = 0,\ 2 < i$. Suppose the scalar stochastic differential equation of the form

$$dx_t = f(x_t, t)dt + g(x_t, t)dB_t,$$

using the definition of the coefficient $k_n(x)$ of the stochastic equation (16), i.e. $\left\langle \frac{(x_\tau - x)^n}{\tau} \right\rangle = k_n(x),\ \tau \to 0$, we have

$$k_1(x) = f(x, t),$$
$$k_2(x) = g^2(x, t),$$

and the higher-order coefficients of the stochastic equation will vanish as a consequence of the Itô differential rule. Thus, the Fokker-Planck equation

$$\dot{p}(x) = -\frac{\partial}{\partial x} f(x, t) p(x) + \frac{1}{2} \frac{\partial^2 g^2(x, t)}{\partial x^2} p(x).$$

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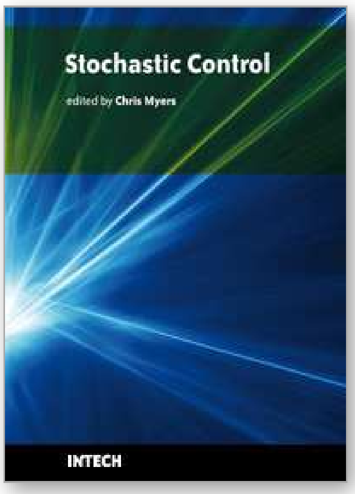

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University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
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