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Timoshenko Beam Theory based Dynamic Modeling of Lightweight Flexible Link Robotic Manipulators

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1. Introduction

In recent years, establishing more and more explicit, complete and accurate dynamic models for the special category of flexible link manipulators has been a formidable challenging and still open problem in robotics research.
This chapter is devoted to a methodological presentation of the application of Timoshenko beam (TB) theory (TBT) concepts to the mathematical description of flexible link robotic manipulators dynamics, as a more refined modeling approach compared to the classical Euler-Bernoulli (EB) theory (EBT) which is the conventionally adopted one.
Compared with the conventional heavy and bulky rigid robots, the flexible link manipulators have their special potential advantages of larger work volume, higher operation speed, greater payload-to-manipulator weight ratio, lower energy consumption, better manoeuvrability and better transportability. However, their utilization incurs a penalty due to elastic deformation and vibration typically associated with the structural flexibility. As a consequence, the motion planning and dynamics modeling of this class of robotic manipulators are apparently made extremely complicated, as well as their tip position control.
The complexity of modeling and control of lightweight flexible manipulators is widely reported in the literature. Detailed discussions can be found in (Kanoh et al., 1986; Baruh & Taikonda, 1989; Book, 1990; Yurkovich, 1992; Book, 1993; Junkins & Kim, 1993; Canudas de Wit et al., 1996; Moallem et al., 2000; Benosman et al., 2002; Robinett et al., 2002; Wang & Gao, 2003; Benosman & Vey, 2004; Dwivedy & Eberhard, 2006, Tokhi & Azad, 2008).
In order to fully exploit the potential advantages offered by these lightweight robot arms, one must explicitly consider the effects of structural link flexibility and properly deal with (active and/or passive) control of vibrational behavior. In this context, it is highly desirable to have an explicit, complete and accurate dynamic model at disposal.
In this chapter, we aim to present the details of our investigations concerned with deriving accurate equations of motion of a flexible link robot arm by the use of the TBT.
In the first part of this work, a brief review of different beam theories and especially that of Timoshenko is given. Then, based on the TBT, the emphasis is essentially set on a detailed description of the different steps, allowing the obtaining of accurate and complete

governing equations of the transversely vibrating motion of an actuated lightweight flexible link robot arm carrying a payload at its free end-point. To display the most relevant aspects of structural properties inherent to the modeled deformable link studied as a beam, important damping mechanisms often ignored, internal structural viscoelasticity (Kelvin-Voigt damping) and external viscous air damping, are included in addition to the TBT effects of cross section shear deformation and rotational inertia.
In the other part of this chapter, an illustrative application case of the above presentation is rigorously highlighted. A new comprehensive dynamic model of a planar single link flexible manipulator considered as a shear deformable TB with internal structural viscoelasticity is proposed. On the basis of the combined Lagrangian-Assumed Modes Method with specific accurate boundary conditions (BCs), the full development details leading to the establishment of a closed form dynamic model, suitable for control purposes, are given.

2. Timoshenko Beam Theory Based Mathematical Modeling

2.1 Brief review of beam theories

A rigorous mathematical model widely used for describing the transverse vibration of beams is based on the TBT (or thick beam theory) (Timoshenko, 1974) developed by Timoshenko in the 1920s. This partial differential equation (PDE) based model is chosen because it is more accurate in predicting the beam's response than the EB beam (EBB) theory (EBBT) (Meirovitch, 1986) one (Aldraihem, 1997; Geist & McLaughlin, 2001; Stephen, 2006; Salarieh & Ghorashi, 2006). Indeed, it has been shown in the literature that the predictions of the TB model are in excellent agreement with the results obtained from the exact elasticity equations and experimental results (Trail-Nash & Collar, 1953; Huang, 1961; Stephen, 1982; Han et al., 1999; Stephen, 2006).
Historically, the first important beam model was the one based on the EBT thin or classical beam theory as a result of the works of the Bernoulli's (Jacob and Daniel) and Euler. This model, established in 1744, includes the strain energy due to the bending and the kinetic energy due to the lateral displacement of the beam. In 1877, Lord Rayleigh improved it by including the effect of rotary inertia in the equations describing the flexural and longitudinal vibrations of beams by showing the importance of this correction especially at high frequency frequencies (Rayleigh, 2003). In 1921 and 1922, Timoshenko proposed another improvement by adding the effect of shear deformation (Timoshenko, 1921; Timoshenko, 1922). He showed, through the example of a simply-supported beam, that the correction due to shear is four times more important than that due to rotary inertia and that the EB and Rayleigh beam equations are special cases of his new result. As a summary, four beam models can be pointed out (Table 1), the EBB and TB models being the most widely used.
As seen above, the TBT accounts for both the effect of rotary inertia and shear deformation, which are neglected when applied to EBBT. The transverse vibration of the beam depends on its geometrical and material properties as well as the external applied torque. The geometrical properties refer mainly to its length ℓ, size and shape of its cross-section such as its area A, moment of inertia I with respect to the central axis of bending, and Timoshenko's shear coefficient k which is a modifying factor ($k<1$) to account for the distribution of shearing stress such that effective shear area is equal to kA. The material properties refer to its density in mass per unit volume ρ, Young's modulus or modulus of elasticity E and shear modulus or modulus of rigidity G.

Beam model \ Effect	*Lateral displacement*	*Bending moment*	*Rotary inertia*	*Shear deformation*
Euler-Bernoulli	+	+	–	–
Rayleigh	+	+	+	–
Shear	+	+	–	+
Timoshenko	+	+	+	+

Table 1. The four beam models with the corresponding effects

2.2 The flexible robotic system: Definitions and variables

The flexible manipulator physical system under consideration is shown in Fig. 1. It consists of a pinned-free or a clamped-free with tip payload (see Fig. 2) planar moving flexible arm which can bend freely in the horizontal plane. The deflection which is the transverse displacement of the link from the X -axis is denoted by $w(x,t)$.

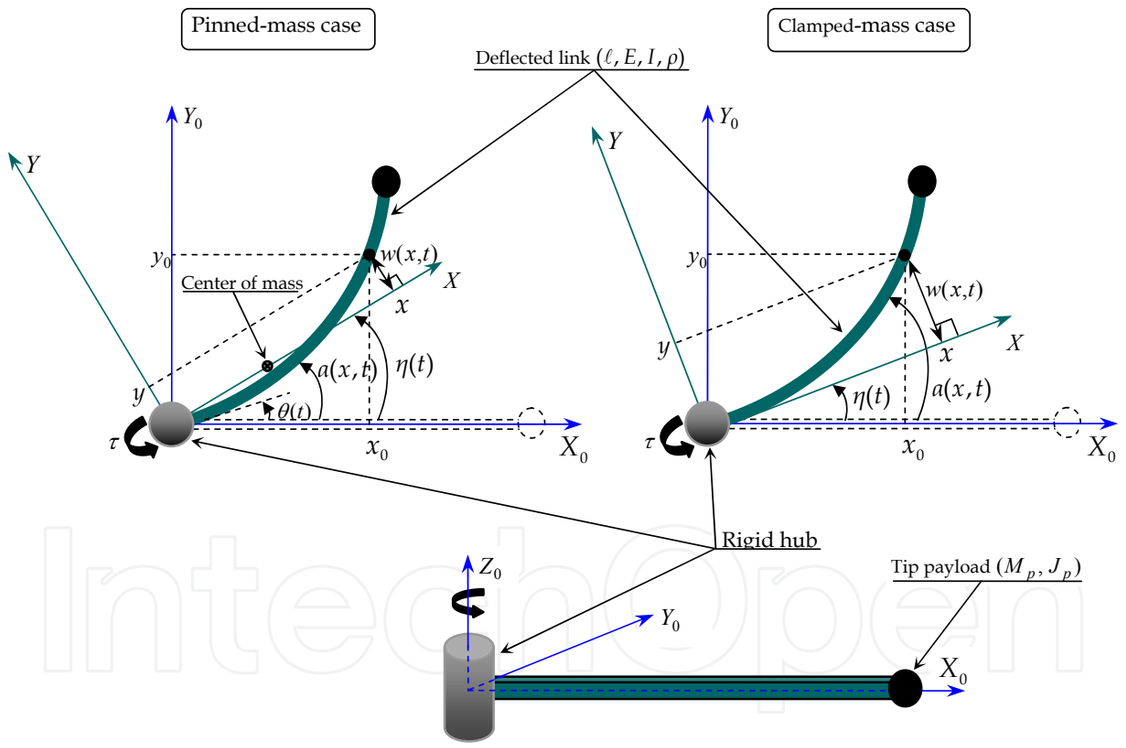

Fig. 1. Pinned and clamped configurations of the considered flexible link manipulator arm

The Fig. 1. is a "top" view of the manipulator in deflection and the axis of rotation of the rigid hub (Z_0) is perpendicular to robot evolution plane. The $X_0 - Y_0$ coordinate frame is the inertial frame of reference. The one indicated by $X - Y$ is a frame of reference that rotates with the overall structure.

For the pinned case (Loudini et al., 2007a; Loudini et al., 2007b), the X -axis is intersecting the center of mass of the whole system. In the clamped case (Loudini et al., 2006), the X -

axis is tangent to the beam at the base (Bellezza et al., 1990).

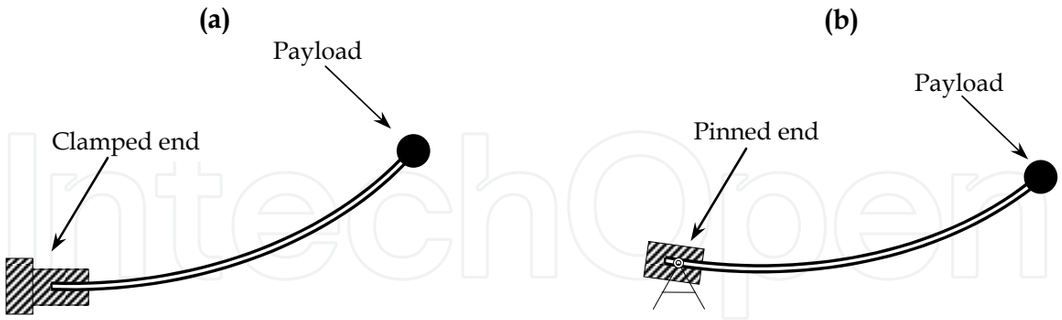

Fig. 2. The two different cases: (a) Clamped-Mass, (b) Pinned-Mass

In Fig. 2., the first case is named Clamped-Mass, meaning that one end is blocked in both angular and vertical direction, and the other end is carrying an inertia load.
The second case is named Pinned-Mass and, as before, it is locked at one end in the vertical direction but free to move in the angular like if it were mounted to a rotary actuator that did not provide a torque, and carrying an inertia load at the other end.
Considering, as usual, the flexible link as a beam, its cross-section height is assumed to be larger than the base. This constrains deflections to occur only in the horizontal plane. Thus, those due to gravity are assumed negligible.
As depicted in Fig. 1, the robot manipulator is essentially composed of a rigid hub, a flexible link and a payload. These three parts are characterized by different physical and mechanical parameters (see the nomenclature at the end of the chapter). In particular, the rotating inertia of the actuating servomotor and the pinning (clamping) rigid hub are modeled as a single hub inertia J_h. The payload is modeled as an end mass M_p with a rotational inertia J_p. $\eta(t)$ being the rotating X-axis angular position, the angular position of the hub, $\theta(t)$ (for the pinned case), and that of a point of the deflected link, $a(x,t)$, are, respectively given, for small deflections, by:

$$\theta(t) = \eta(t) + \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} \quad \text{(pinned case)} \tag{1}$$

$$a(x,t) = \eta(t) + \frac{w(x,t)}{x} \tag{2}$$

2.3 Derivation of the governing equations of motion

The kinematics of deformation of an element of the deflected link with width dx at position x are shown in Fig. 3. Due to the effect of shear, the original rectangular element changes its shape to somewhat like a parallelogram with its sides slightly curved.

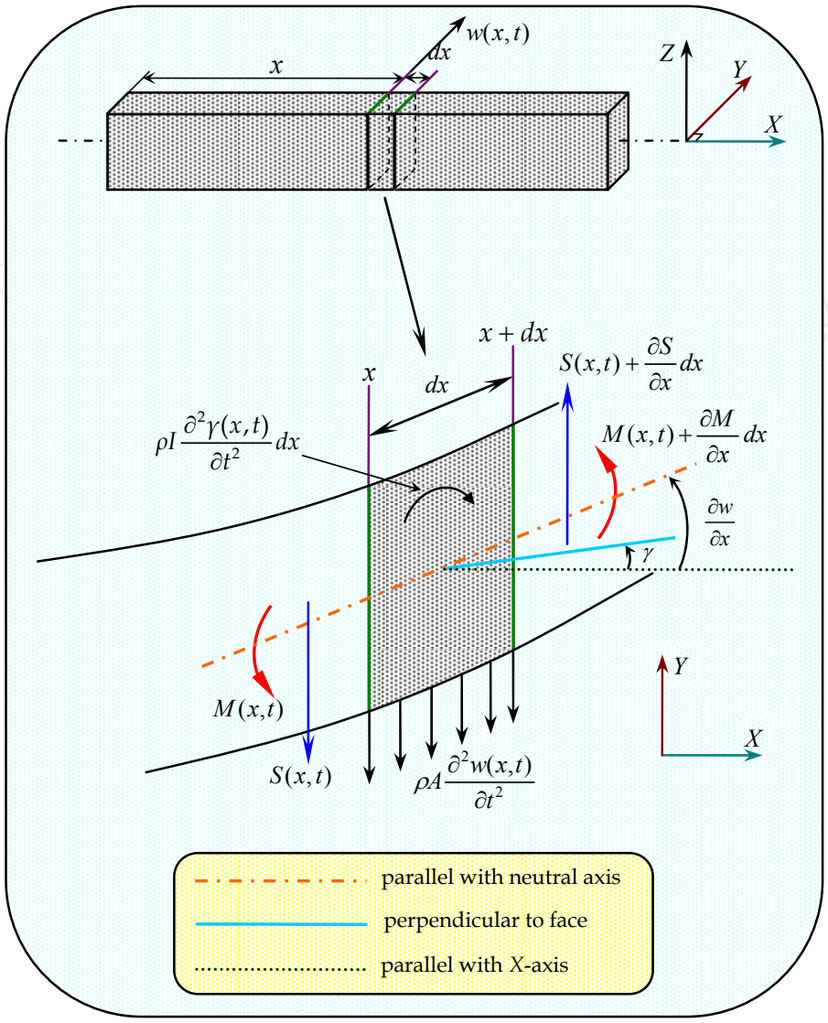

Fig. 3. Kinematics of deformation of a bending element

This element undergoes a shearing force $S(x,t)$ and a bending moment $M(x,t)$. On the opposite side, which corresponds to a position $x+dx$, the shearing force ($S+dS$) is

$$S(x+dx,t) = S(x,t) + \frac{\partial S(x,t)}{\partial x} dx \tag{3}$$

Likewise the moment force ($M+dM$) at the position $x+dx$ is

$$M(x+dx,t) = M(x,t) + \frac{\partial M(x,t)}{\partial x} dx \tag{4}$$

Note that the total deflection is due to both bending and shear forces, so that the shear angle $\sigma(x,t)$ (or loss of slope) is equal to the slope of centerline (neutral axis) $\frac{\partial w(x,t)}{\partial x}$ less slope of bending $\gamma(x,t)$:

$$\sigma(x,t) = \frac{\partial w(x,t)}{\partial x} - \gamma(x,t) \tag{5}$$

The shear force S is given by

$$S(x,t) = kAG\sigma(x,t) = kAG\left[\frac{\partial w(x,t)}{\partial x} - \gamma(x,t)\right] \tag{6}$$

By considering the "standard linear solid (SLS) model" or Zener model (Zener, 1965), with the stress-strain law given by

$$\nu + C_D \frac{\partial \nu}{\partial t} = E\varepsilon + K_D \frac{\partial \varepsilon}{\partial t} \tag{7}$$

and assuming linear variations of strain and stress across the beam depth, the total moment obtained by integrating first moment of stress across the beam cross section is (Baker et al., 1967):

$$M(x,t) = \left(1 + C_D \frac{\partial}{\partial t}\right) M_0 = I\left(E + K_D \frac{\partial}{\partial t}\right) \gamma_x(x,t) \tag{8}$$

The total internal moment (bending and damping) M is then given by (Banks & Inman, 1991; Banks et al., 1994)

$$M(x,t) = EI \frac{\partial \gamma(x,t)}{\partial x} + K_D I \frac{\partial^2 \gamma(x,t)}{\partial x \partial t} \tag{9}$$

The equation of motion of the studied single link elastic robot arm can be derived by considering both the equilibrium of the moments and the forces.
Taking moments as positive in the counter-clockwise direction, their summation with disregarding the second order term of dx, yields the relation between the spatial change in the bending moment and the shear force

$$\frac{\partial M(x,t)}{\partial x} = -S(x,t) + \rho I \frac{\partial^2 \gamma(x,t)}{\partial t^2} \tag{10}$$

where the term $\rho I \frac{\partial^2 \gamma(x,t)}{\partial t^2}$ stands for the distributed rotational inertia given by the product of the mass moment of inertia of the cross section and the angular acceleration.

The relation that fellows balancing forces is

$$\frac{\partial S(x,t)}{\partial x} - A_D \frac{\partial w(x,t)}{\partial t} = \rho A \frac{\partial^2 w(x,t)}{\partial t^2} \tag{11}$$

where the terms $A_D \frac{\partial w(x,t)}{\partial t}$, $\rho A \frac{\partial^2 w(x,t)}{\partial t^2}$ represent, respectively, the air resistance force and the distributed transverse inertial force.

Substitution of (6) and (9) into (10) and likewise (6) into (11) yields the two coupled equations of the damped TB motion:

$$K_D I \frac{\partial^3 \gamma(x,t)}{\partial x^2 \partial t} + EI \frac{\partial^2 \gamma(x,t)}{\partial x^2} + kAG\left(\frac{\partial w(x,t)}{\partial x} - \gamma\right) - \rho I \frac{\partial^2 \gamma(x,t)}{\partial t^2} = 0 \tag{12}$$

$$kAG\left(\frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial \gamma(x,t)}{\partial x}\right) - \rho A \frac{\partial^2 w(x,t)}{\partial t^2} - A_D \frac{\partial w(x,t)}{\partial t} = 0 \tag{13}$$

If the damping effects terms are suppressed, the classical set of two coupled PDEs developed by Timoshenko (Timoshenko, 1921; Timoshenko, 1922) arises:

$$EI \frac{\partial^2 \gamma(x,t)}{\partial x^2} + kAG\left(\frac{\partial w(x,t)}{\partial x} - \gamma\right) - \rho I \frac{\partial^2 \gamma(x,t)}{\partial t^2} = 0 \tag{14}$$

$$kAG\left(\frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial \gamma(x,t)}{\partial x}\right) - \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{15}$$

The modeled beam cross-sectional area and density being uniform, equations (14) and (15) can be easily decoupled as follows:

$$\begin{aligned} &K_D I \frac{\partial^5 w(x,t)}{\partial x^4 \partial t} - \frac{K_D I \rho}{KG} \frac{\partial^5 w(x,t)}{\partial x^2 \partial t^3} + EI \frac{\partial^4 w(x,t)}{\partial x^4} - \rho I\left(1 + \frac{E}{KG} + \frac{K_D A_D}{\rho KAG}\right) \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + \dots \\ &\dots \frac{\rho^2 I}{KG} \frac{\partial^4 w(x,t)}{\partial t^4} - \frac{EIA_D}{kAG} \frac{\partial^3 w(x,t)}{\partial x^2 \partial t} + \frac{\rho I A_D}{kAG} \frac{\partial^3 w(x,t)}{\partial t^3} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + A_D \frac{\partial w(x,t)}{\partial t} = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} &K_D I \frac{\partial^5 \gamma(x,t)}{\partial x^4 \partial t} - \frac{K_D I \rho}{KG} \frac{\partial^5 \gamma(x,t)}{\partial x^2 \partial t^3} + EI \frac{\partial^4 \gamma(x,t)}{\partial x^4} - \rho I\left(1 + \frac{E}{KG} + \frac{K_D A_D}{\rho KAG}\right) \frac{\partial^4 \gamma(x,t)}{\partial x^2 \partial t^2} + \dots \\ &\dots \frac{\rho^2 I}{KG} \frac{\partial^4 \gamma(x,t)}{\partial t^4} - \frac{EIA_D}{kAG} \frac{\partial^3 \gamma(x,t)}{\partial x^2 \partial t} + \frac{\rho I A_D}{kAG} \frac{\partial^3 \gamma(x,t)}{\partial t^3} + \rho A \frac{\partial^2 \gamma(x,t)}{\partial t^2} + A_D \frac{\partial \gamma(x,t)}{\partial t} = 0 \end{aligned} \tag{17}$$

Similar to the those established in (De Silva, 1976; Sooraksa & Chen, 1998), equation (16) is the fifth order TB homogeneous linear PDE with internal and external damping effects expressing the deflection $w(x,t)$.

We have added to this equation the following initial and pinned (clamped)-mass boundary conditions (Loudini et al., 2007a, Loudini et al., 2006):

Initial conditions:
$$w(x,0) = w_0 \ , \ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = \dot{w}_0 \tag{18}$$

Pinned end:
$$w(0,t) = 0 \ , \ \left[M(x,t) - J_h \frac{\partial^3 w(x,t)}{\partial x \partial t^2} \right]_{x=0} = 0 \tag{19}$$

Clamped end:
$$w(0,t) = 0 \ , \ \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0 \tag{20}$$

Free end with payload mass:
$$\begin{cases} \left[\dfrac{\partial M(x,t)}{\partial x} - M_p \dfrac{\partial^2 w(x,t)}{\partial t^2} \right]_{x=\ell} = 0 \\ \left[M(x,t) + J_p \dfrac{\partial^3 w(x,t)}{\partial x \partial t^2} \right]_{x=\ell} = 0 \end{cases} \tag{21}$$

The classical fourth order TB PDE is retrieved if the damping effects terms are suppressed:

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} - \rho I \left(1 + \frac{E}{KG} \right) \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{KG} \frac{\partial^4 w(x,t)}{\partial t^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{22}$$

If the effect due to the rotary inertia is neglected, we are led to the shear beam (SB) model (Morris, 1996; Han et al., 1999):

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} - \frac{\rho I E}{KG} \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{23}$$

but, if the one due to shear distortion is the neglected one, the Rayleigh beam equation (Han et al., 1999; Rayleigh, 2003) arises:

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} - \rho I \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{24}$$

Moreover, if both the rotary inertia and shear deformation are neglected, then the governing equation of motion reduces to that based on the classical EBT (Meirovitch, 1986) given by

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{25}$$

If the above included damping effects are associated to the EBB, the corresponding PDE is

$$K_D I\frac{\partial^5 w(x,t)}{\partial x^4\partial t}+EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}+A_D\frac{\partial w(x,t)}{\partial t}=0 \tag{26}$$

The resolution of the PDE with mixed derivative terms (16) is a complex mathematical problem. Among the few methods existing in the literature, we cite the following approaches with some representative works: the finite element method (Kapur, 1966; Hoa, 1979; Kolberg 1987), the Galerkin method (Wang and Chou, 1998; Dadfarnia et al., 2005), the Rayleigh-Ritz method (Oguamanam and Heppler, 1996), the Laplace transform method resulting in an integral form solution (Boley & Chao, 1955; Wang & Guan, 1994; Ortner & Wagner, 1996), and the eigenfunction expansion method, also referred to as the series or modal expansion method (Anderson, 1953; Dolph, 1954; Huang, 1961; Ekwaro-Osire et al., 2001; Loudini et al. 2006; Loudini et al. 2007a; Loudini et al. 2007b).

In the latter one, $w(x,t)$ can take the following expanded separated form which consists of an infinite sum of products between the chosen transverse deflection eigenfunctions or mode shapes $W_n(x)$, that must satisfy the pinned (clamped)-free (mass) BCs, and the time-dependant modal generalized coordinates $\delta_n(t)$:

$$w(x,t)=\sum_{n=1}^{\infty}W_n(x)\delta_n(t) \tag{27}$$

2.4 Dynamic model deriving procedure

In order to obtain a set of ordinary differential equations (ODEs) of motion to adequately describe the dynamics of the flexible link manipulator, the Hamilton's or Lagrange's approach combined with the Assumed Modes Method (AMM) (Fraser & Daniel, 1991; Loudini et al. 2006; Loudini et al. 2007a; Loudini et al. 2007b; Tokhi & Azad, 2008) can be used.

According to the Lagrange's method, a dynamic system completely located by n generalized coordinates q_i must satisfy n differential equations of the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}+\frac{\partial D}{\partial \dot{q}_i}=F_i,\ i=0,1,2,\cdots \tag{28}$$

where L is the so-called Lagrangian given by

$$L=T-U \tag{29}$$

T represents the kinetic energy of the modeled system and U its potential energy. Also, in (28) D is the Rayleigh's dissipation function which allows dissipative effects to be included, and F_i is the generalized external force acting on the corresponding coordinate q_i.

Theoretically there are infinite number of ODEs, but for practical considerations, such as boundedness of actuating energy and limitation of the actuators and the sensors working frequency range, it is more reasonable to truncate this number at a finite one n (Cannon & Schmitz, 1984; Kanoh & Lee, 1985; Qi & Chen, 1992).

The total kinetic energy of the robot flexible link and its potential energy due to the internal bending moment and the shear force are, respectively, given by (Macchelli & Melchiorri, 2004; Loudini et al. 2006; Loudini et al. 2007a; Loudini et al. 2007b):

$$T = \frac{1}{2}\int_0^{\ell} \rho A \left[\frac{\partial w(x,t)}{\partial t}\right]^2 dx + \frac{1}{2}\int_0^{\ell} \rho I \left[\frac{\partial \gamma(x,t)}{\partial t}\right]^2 dx \tag{30}$$

$$U = \frac{1}{2}\int_0^{\ell} EI \left[\frac{\partial \gamma(x,t)}{\partial x}\right]^2 dx + \frac{1}{2}\int_0^{\ell} KAG\left[\sigma(x,t)\right]^2 dx \tag{31}$$

The dissipated energy due to the damping effects can be written as (Krishnan & Vidyasagar, 1988; Loudini et al. 2006; Loudini et al. 2007a; Loudini et al. 2007b):

$$D = \frac{1}{2}\int_0^{\ell} A_D \left[\frac{\partial w(x,t)}{\partial t}\right]^2 dx + \frac{1}{2}\int_0^{\ell} K_D I \left[\frac{\partial^3 w(x,t)}{\partial x^2 \partial t}\right]^2 dx \tag{32}$$

Substituting these energies expressions into (28) accordingly and using the transverse deflection separated form (27), we can derive the desired dynamic equations of motion in the mass (B), damping (H), Coriolis and centrifugal forces (N) and stiffness (K) matrix familiar form:

$$B\frac{d^2 q(t)}{dt^2} + H\frac{dq(t)}{dt} + N\left(q(t), \dot{q}(t)\right) + Kq(t) = F(t) \tag{33}$$

with $q(t) = \left[\theta(t) \quad \delta_1(t) \quad \delta_2(t) \quad \cdots \quad \delta_n(t)\right]^T$; $F(t) = \left[\tau \quad 0 \quad 0 \quad \cdots \quad 0\right]^T$.

If we disregard some high order and nonlinear terms, under reasonable assumptions, the matrix differential equation in (33) could be easily represented in a state-space form as

$$\begin{cases} \dot{z}(t) = A_z z(t) + B_z u(t) \\ y(t) = C_z z(t) \end{cases} \tag{34}$$

with $u(t)=\begin{bmatrix}\tau & 0 & \cdots & 0\end{bmatrix}^T$; $z(t)=\begin{bmatrix}\theta(t) & \delta_1(t) & \cdots & \delta_n(t) & \dot{\theta}(t) & \dot{\delta}_1(t) & \cdots & \dot{\delta}_n(t)\end{bmatrix}^T$.

Solving the state-space matrices gives the vector of states $z(t)$, that is, the angular displacement, the modal amplitudes and their velocities.

3. A Special Case Study: Comprehensive Dynamic Modeling of a Flexible Link Manipulator Considered as a Shear Deformable Timoshenko Beam

In this second part of our work, we present a novel dynamic model of a planar single-link flexible manipulator considered as a tip mass loaded pinned-free shear deformable beam. Using the classical TBT described in section 2 and including the Kelvin-Voigt structural viscoelastic effect (Christensen, 2003), the lightweight robotic manipulator motion governing PDE is derived. Then, based on the Lagrange's principle combined with the AMM, a dynamic model suitable for control purposes is established.

3.1 System description and motion governing equation

The considered physical system is shown in Fig. 4. The basic deriving procedure to obtain the motion governing equation has been described in the previous section, and so only an outline giving the main steps is presented here.

The effect of rotary inertia being neglected in this case study, equation (10) expressing the equilibrium of the moments becomes:

$$\frac{\partial M(x,t)}{\partial x}=-S(x,t) \tag{35}$$

The relation that fellows balancing forces is

$$\frac{\partial S(x,t)}{\partial x}=\rho A\frac{\partial^2 w(x,t)}{\partial t^2} \tag{36}$$

Substitution of 6 and 9 into 35 and likewise 6 into 36 yields the two coupled equations of the damped SB motion:

$$K_D I\frac{\partial^3 \gamma(x,t)}{\partial x^2\partial t}+EI\frac{\partial^2 \gamma(x,t)}{\partial x^2}+kAG\left[\frac{\partial w(x,t)}{\partial x}-\gamma(x,t)\right]=0 \tag{37}$$

$$kAG\left[\frac{\partial^2 w(x,t)}{\partial x^2}-\frac{\partial \gamma(x,t)}{\partial x}\right]-\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{38}$$

Equations 37 and 38 can be easily decoupled to obtain the fifth order SB homogeneous linear PDEs with internal damping effect expressing the deflection $w(x,t)$ and the slope of bending $\gamma(x,\mathrm{t})$

$$K_D I\frac{\partial^5 w(x,t)}{\partial x^4 \partial t}-\frac{\rho K_D I}{KG}\frac{\partial^5 w(x,t)}{\partial x^2 \partial t^3}+EI\frac{\partial^4 w(x,t)}{\partial x^4}-\frac{\rho EI}{KG}\frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{39}$$

$$K_D I\frac{\partial^5 \gamma(x,t)}{\partial x^4 \partial t}-\frac{\rho K_D I}{KG}\frac{\partial^5 \gamma(x,t)}{\partial x^2 \partial t^3}+EI\frac{\partial^4 \gamma(x,t)}{\partial x^4}-\frac{\rho EI}{KG}\frac{\partial^4 \gamma(x,t)}{\partial x^2 \partial t^2}+\rho A\frac{\partial^2 \gamma(x,t)}{\partial t^2}=0 \tag{40}$$

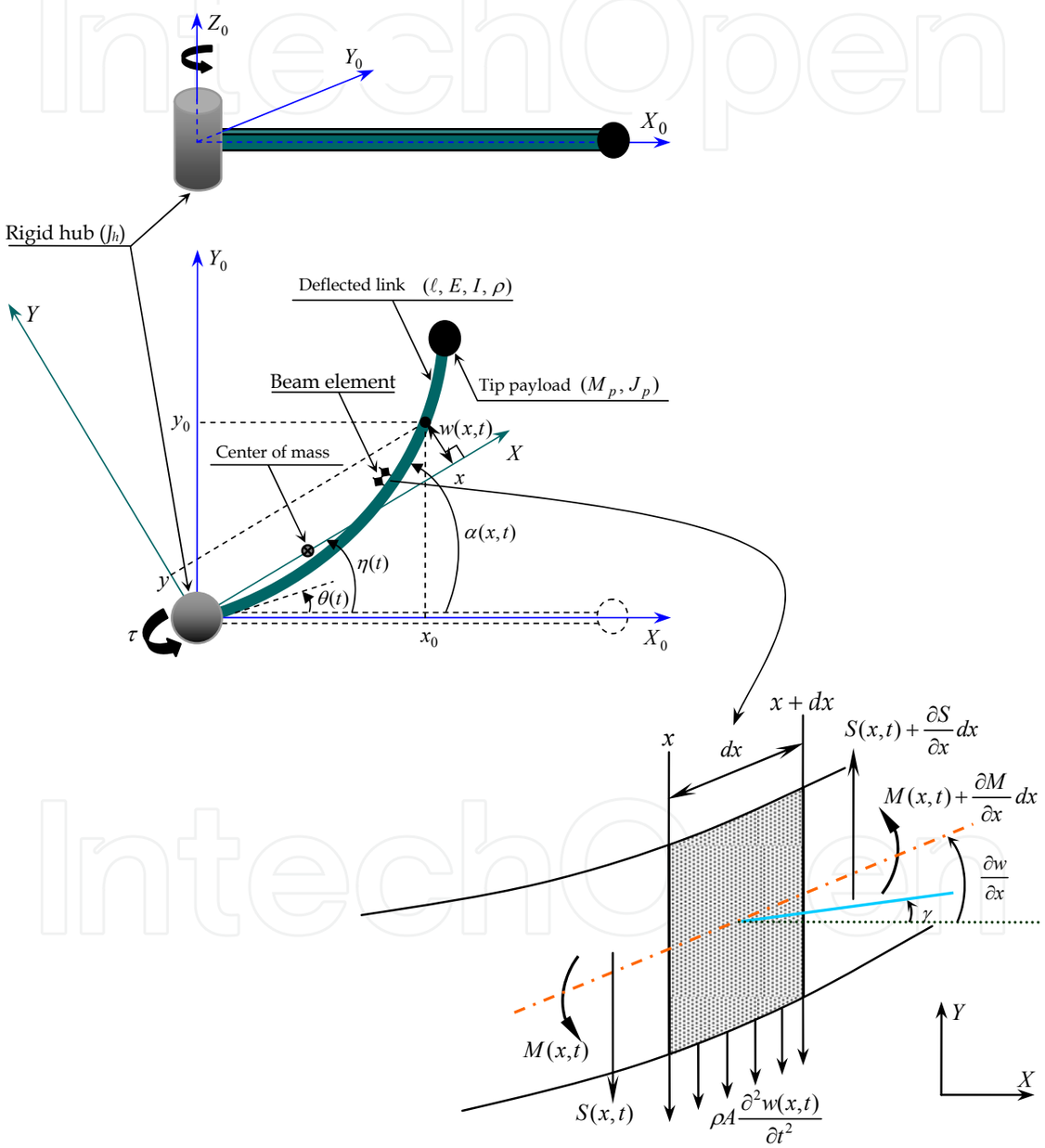

Fig. 4. Physical configuration and kinematics of deformation of a bending element of the studied flexible robot manipulator considered as a shear deformable beam

We affect to the equation (39) the same initial and pinned-mass boundary conditions, given by equations 18, 19, and 21, with taking into account the result established by (Wang & Guan, 1994; Loudini et al., 2007b) about the very small influence of the tip payload inertia on the flexible manipulator dynamics:

Initial conditions: $$w(x,0) = w_0 \,, \;\; w_t(x,t)\big|_{t=0} = \dot{w}_0 \tag{41}$$

BCs at the pinned end (root of the link):

$$w(x,t)\big|_{x=0} = 0 \text{ : zero average translational displacement} \tag{42}$$

$$M(x,t)\big|_{x=0} = J_h \left.\frac{\partial^3 w(x,t)}{\partial x \partial t^2}\right|_{x=0} \text{ : balance of bending moments} \tag{43}$$

BCs at the mass loaded free end:

$$M(x,t)\big|_{x=\ell} = 0 \text{ : zero average of bending moments} \tag{44}$$

$$\left.\frac{\partial M(x,t)}{\partial x}\right|_{x=\ell} = M_p \left.\frac{\partial^2 w(x,t)}{\partial t^2}\right|_{x=\ell} \text{ : balance of shearing forces} \tag{45}$$

The classical fourth order SB PDEs are retrieved if the damping effect term is suppressed:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} - \frac{\rho EI}{KG}\frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{46}$$

$$EI\frac{\partial^4 \gamma(x,t)}{\partial x^4} - \frac{\rho EI}{KG}\frac{\partial^4 \gamma(x,t)}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 \gamma(x,t)}{\partial t^2} = 0 \tag{47}$$

Moreover, if shear deformation is neglected, then the governing equation of motion reduces to that based on the classical EBT, given by 25.

If the above included damping effect is associated to the EBB, the corresponding PDE is

$$K_D I\frac{\partial^5 w(x,t)}{\partial x^4 \partial t} + EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{48}$$

To solve the PDEs with mixed derivative terms (39) and (40), we have tried to apply the classical AMM which is well known as a computationally efficient scheme that separates the mode functions from the shape functions.
The forms of equations (39) and (40) being identical, $w(x,t)$ and $\gamma(x,t)$ are assumed to

share the same time-dependant modal generalized coordinate $\delta(t)$ under the following separated forms with the respective mode shape functions (eigenfuntions) $\Phi(x)$ and $\Psi(x)$ that must satisfy the pinned-free (mass) BCs:

$$\begin{aligned} w(x,t) &= \Phi(x)\delta(t) \\ \gamma(x,t) &= \Psi(x)\delta(t) \end{aligned} \tag{49}$$

Unfortunately, the application of 49 has not been possible to derive the mode shapes expressions. This is due to the unseparatability of some terms of 39 and 40.
To find a way to solve the problem, we have based our investigations on the result pointed out in (Gürgöze et al., 2007). In this work, it has been established that the characteristic equation of a visco-elastic EBB i.e., a Kelvin-Voigt model (given in our chapter by 48), is formally the same as the frequency equation of the cantilevered elastic beam (the EB modeled by 25). Thus, we can assume that the damping effect affects only the modal function $\delta(t)$. So, the mode shape is that of the SB model (46, 47).

Applying the AMM to the PDEs 46 and 47, we obtain

$$EI\Phi^{iv}(x)\delta(t) - \frac{\rho EI}{KG}\Phi''(x)\ddot{\delta}(t) + \rho A\Phi(x)\ddot{\delta}(t) = 0 \tag{50}$$

$$EI\Psi^{iv}(x)\delta(t) - \frac{\rho EI}{KG}\Psi''(x)\ddot{\delta}(t) + \rho A\Psi(x)\ddot{\delta}(t) = 0 \tag{51}$$

Separating the functions of time, t, and space x:

$$\frac{\ddot{\delta}(t)}{\delta(t)} = -\frac{\Phi^{iv}(x)}{\frac{\rho A}{EI}\Phi(x) - \frac{\rho}{KG}\Phi^{ii}(x)} = -\frac{\Psi^{iv}(x)}{\frac{\rho A}{EI}\Psi(x) - \frac{\rho}{KG}\psi^{ii}(x)} = \text{constant} = -\lambda \tag{52}$$

The differential equation for the temporal modal generalized coordinate is

$$\ddot{\delta}(t) + \lambda\delta(t) = 0 \tag{53}$$

Its general solution is assumed to be in the following forms:

$$\delta(t) = De^{j\omega t} + \bar{D}e^{-j\omega t} = F\cos(\omega t + \varphi) \tag{54}$$

where

$$\lambda = \omega^2 \tag{55}$$

The constants D and its complex conjugate $\bar{D}$ (or F and the phase ϕ) are determined from the initial conditions. The natural frequency ω is determined by solving the spatial problem given by

$$\begin{aligned}\Phi^{iv}(x)+\frac{\rho}{KG}\omega^2\Phi^{ii}(x)-\frac{\rho A}{EI}\omega^2\Phi(x)=0\\ \Psi^{iv}(x)+\frac{\rho}{KG}\omega^2\Psi^{ii}(x)-\frac{\rho A}{EI}\omega^2\Psi(x)=0\end{aligned} \tag{56}$$

The solutions of 56 can be written in terms of sinusoidal and hyperbolic functions

$$\begin{aligned}\Phi(x)=C_1\sin ax+C_2\cos ax+C_3\sinh bx+C_4\cosh bx\\ \Psi(x)=D_1\sin ax+D_2\cos ax+D_3\sinh bx+D_4\cosh bx\end{aligned} \tag{57}$$

where

$$a=\sqrt{\frac{\rho}{2KG}\omega^2+\sqrt{\left(\frac{\rho}{2KG}\omega^2\right)^2+\frac{\rho A}{EI}\omega^2}}\quad;\quad b=\sqrt{-\frac{\rho}{2KG}\omega^2+\sqrt{\left(\frac{\rho}{2KG}\omega^2\right)^2+\frac{\rho A}{EI}\omega^2}} \tag{58}$$

The constants $C_k, D_k; k=\overline{1,4}$ of 57 are determined through the BCs 42-45 rewritten on the basis of 49, 53 and 55 as follows:

$$\Phi(0)=0 \tag{59}$$

$$\Psi'(0)=-\frac{J_h}{EI}\omega^2\Phi'(0)=-J\Phi'(0) \tag{60}$$

$$\Psi'(\ell)=0 \tag{61}$$

$$\Phi'(\ell)-\Psi(\ell)=\frac{M_p}{KAG}\omega^2\Phi(\ell)=M\Phi(\ell) \tag{62}$$

By applying 59-62 to 57, we find these relations

$$C_2=-C_4 \tag{63}$$

$$aD_1+bD_3=-aJC_1-bJC_3 \tag{64}$$

$$aD_1\cos a\ell-aD_2\sin a\ell+bD_3\cosh b\ell+bD_4\sinh b\ell=0 \tag{65}$$

$$\left(C_1 a - C_2 M - D_2\right)\cos a\ell - \left(C_2 a + C_1 M + D_1\right)\sin a\ell + \left(C_3 b - C_4 M - D_4\right)\cosh b\ell + \cdots$$
$$\cdots\left(C_4 b - C_3 M - D_3\right)\sinh b\ell = 0 \tag{66}$$

The relations between the unknown constants C_k and D_k are obtained by substituting 57 into 38:

$$D_1 = \frac{R-a^2}{a}C_2;\ D_2 = -\frac{R-a^2}{a}C_1;\ D_3 = \frac{R+b^2}{b}C_4;\ D_4 = \left(\frac{R+b^2}{b}\right)C_3 \tag{67}$$

or

$$C_1 = -\frac{a}{\left(R-a^2\right)}D_2;\ C_2 = \frac{a}{\left(R-a^2\right)}D_1;\ C_3 = \frac{b}{\left(R+b^2\right)}D_4;\ C_4 = \frac{b}{\left(R+b^2\right)}D_3 \tag{68}$$

where $R = \dfrac{\rho\omega^2}{KG}$.

From 63 and 67, we obtain

$$D_3 = -\frac{a\left(R+b^2\right)}{b\left(R-a^2\right)}D_1 = D_{31}D_1 \tag{69}$$

From some combinations of 63-69, we find the relations

$$C_2 = \left[-\frac{\dfrac{a\left(R+b^2\right)}{b\left(R-a^2\right)}\sinh b\ell - \sin a\ell}{\cos a\ell - \dfrac{R+b^2}{R-a^2}\cosh b\ell + \dfrac{\left(a^2+b^2\right)\left(R+b^2\right)}{bJ\left(R-a^2\right)}\sinh b\ell}\right]C_1 = C_{21}C_1 \tag{70}$$

$$C_3 = \left[\frac{\left(\dfrac{R}{a} - MC_{21}\right)\cos a\ell - \left(\dfrac{R}{a}C_{21} + M\right)\sin a\ell + MC_{21}\cosh b\ell + \dfrac{R}{b}C_{21}\sinh b\ell}{M\sinh b\ell - \dfrac{R}{b}\cosh b\ell}\right]C_1 = C_{31}C_1 \tag{71}$$

$$D_2 = \left[\frac{\cos a\ell - \dfrac{R+b^2}{R-a^2}\cosh b\ell + \dfrac{\left(a^2+b^2\right)\left(R+b^2\right)}{bJ\left(R-a^2\right)}\sinh b\ell}{\dfrac{a\left(R+b^2\right)}{b\left(R-a^2\right)}\sinh b\ell - \sin a\ell} \right] D_1 = D_{21}D_1 \tag{72}$$

$$D_4 = \left[\begin{array}{c} \dfrac{aMD_{21}-R}{R-a^2}\sin a\ell - \dfrac{RD_{21}+aM}{R-a^2}\cos a\ell + \left(\dfrac{aR}{b\left(R-a^2\right)} + \dfrac{bM}{R+b^2}\right)\sinh b\ell + \cdots \\ \cdots \dfrac{aM}{R-a^2}\cosh b\ell \\ \hline \dfrac{R}{R+b^2}\cosh b\ell \end{array} \right] D_1 \tag{73}$$

$$= D_{41}D_1$$

Replacing (63) and (69)-(73) into (57), we obtain

$$\begin{aligned} \Phi(x) &= C_1\left[\sin ax + C_{21}\left(\cos ax - \cosh bx\right) + C_{31}\sinh bx\right] \\ \Psi(x) &= D_1\left[\sin ax + D_{21}\cos ax + D_{31}\sinh bx + D_{41}\cosh bx\right] \end{aligned} \tag{74}$$

In order to establish the frequency equation, we rewrite the equations 63-66 as fellows

$$C_2 + C_4 = 0 \tag{75}$$

$$aJC_1 + \left(R-a^2\right)C_2 + bJC_3 + \left(R+b^2\right)C_4 = 0 \tag{76}$$

$$\underbrace{\left(R-a^2\right)\sin a\ell}_{CF_1}C_1 + \underbrace{\left(R-a^2\right)\cos a\ell}_{CF_2}C_2 + \underbrace{\left(R+b^2\right)\sinh b\ell}_{CF_3}C_3 + \underbrace{\left(R+b^2\right)\cosh b\ell}_{CF_4}C_4 = 0 \tag{77}$$

$$\begin{aligned} &\underbrace{\left(\frac{R}{a}\cos a\ell - M\sin a\ell\right)}_{CF_5}C_1 + \underbrace{\left(-M\cos a\ell - \frac{R}{a}\sin a\ell\right)}_{CF_6}C_2 + \underbrace{\left(-\frac{R}{b}\cosh b\ell - M\sinh b\ell\right)}_{CF_7}C_3 + \cdots \\ &\cdots\underbrace{\left(-\frac{R}{b}\sinh b\ell - M\cosh b\ell\right)}_{CF_8}C_4 = 0 \end{aligned} \tag{78}$$

Consider the coefficients of the four equations as a matrix C given by

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ aJ & R-a^2 & bJ & R+b^2 \\ CF_1 & CF_2 & CF_3 & CF_4 \\ CF_5 & CF_6 & CF_7 & CF_8 \end{bmatrix} \tag{79}$$

In order that solutions other than zero may exist, the determinant of C must me null. This leads to the frequency equation

$$\begin{aligned} -\frac{a}{b}RJ\left(R+b^2\right)+\left(a^2+b^2\right)\left[MaJ+\frac{R}{a}\left(R+b^2\right)\right]\cos a\ell\sinh b\ell+\cdots \\ \cdots\left(a^2+b^2\right)\left[MbJ+\frac{R}{b}\left(R-a^2\right)\right]\sin a\ell\cosh b\ell+\cdots \\ \cdots\left[RJ\left(a^2+b^2\right)+\frac{b}{a}RJ\left(R-a^2\right)-M\left(a^2+b^2\right)^2\right]\sin a\ell\sinh b\ell-\cdots \\ \cdots J\left[\frac{a}{b}\left(R-a^2\right)+\frac{b}{a}\left(R+b^2\right)\right]\cos a\ell\cosh b\ell=0 \end{aligned} \tag{80}$$

3.2 Derivation of the dynamic model

As explained before, the energetic Lagrange's principle is adopted.
The total kinetic energy is given by

$$T = T_\ell + T_h + T_p \tag{81}$$

where T_h, T_ℓ and T_p are the kinetic energies associated to, respectively, the rigid hub, the flexible link, and the payload:

$$T_h = \frac{1}{2}J_h\dot{\theta}^2(t) \tag{82}$$

$$T_\ell = \frac{1}{2}\int_0^\ell \rho A\left\{\left[x\dot{\theta}(t)+\frac{\partial w(x,t)}{\partial t}\right]^2+\left[\dot{\theta}(t)w(x,t)\right]^2\right\}dx \tag{83}$$

$$T_p = \frac{1}{2}M_p\left\{\left[\left(x\dot{\theta}(t)+\frac{\partial w(x,t)}{\partial t}\right)_{x=\ell}\right]^2+\left[\dot{\theta}(t)w(\ell,t)\right]^2\right\}+\frac{1}{2}J_p\left\{\left[\dot{\theta}(t)+\frac{\partial}{\partial t}\left(\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=\ell}\right)\right]^2\right\} \tag{84}$$

The potential energy of the system, U, can be written as

$$U = \frac{1}{2}\int_0^\ell EI\left[\frac{\partial \gamma(x,t)}{\partial x}\right]^2 dx + \frac{1}{2}\int_0^\ell KAG\left[\frac{\partial w(x,t)}{\partial x} - \gamma(x,t)\right]^2 dx \tag{85}$$

The dissipated energy D may be written as

$$D = \frac{1}{2}\int_0^\ell K_D I\left[\frac{\partial^3 w(x,t)}{\partial x^2 \partial t}\right]^2 dx \tag{86}$$

Using, for ease of manipulation, the following notations and substitutions

$$\Phi_{i\ell} = \Phi_i(\ell)\ ;\ \Phi_{i\ell}{}^2 = \Phi_i{}^2(\ell)\ ;\ \Phi'_{i\ell} = \Phi'_i(\ell)\ ;\ \Phi'_{i\ell}{}^2 = \Phi'^2_i(\ell);\ \Gamma_{1_i} = \int_0^\ell \Phi_i{}^2(x)dx\ ;\ \Gamma_{1_{12}} = \int_0^\ell \Phi_1(x)\Phi_2(x)dx;$$

$$\Gamma_{2_i} = \int_0^\ell \Psi_i^2(x)dx\ ;\ \Gamma_{2_{12}} = \int_0^\ell \Psi_1(x)\Psi_2(x)dx;\ \Gamma_{3_i} = \int_0^\ell \Phi'^2_i(x)dx\ ;\ \Gamma_{3_{12}} = \int_0^\ell \Phi'_1(x)\Phi'_2(x)dx;$$

$$\Gamma_{4_i} = \int_0^\ell \Psi_i'^2(x)dx\ ;\ \Gamma_{4_{12}} = \int_0^\ell \Psi'_1(x)\Psi'_2(x)dx\ ;\ \Gamma_{5_i} = \int_0^\ell \Phi''^2_i(x)dx\ ;\ \Gamma_{5_{12}} = \int_0^\ell \Phi''_1(x)\Phi''_2(x)dx\ ;\ \Gamma_{6_i} = \int_0^\ell x\Phi_i(x)dx;$$

$$\Gamma_{7_i} = \int_0^\ell \Phi'_i(x)\Psi_i(x)dx\ ;\ \Gamma_{7_{12}} = \int_0^\ell \Phi'_1(x)\Psi_2(x)dx\ ;\ \Gamma_{7_{21}} = \int_0^\ell \Phi'_2(x)\Psi_1(x)dx.$$

we obtain

$$\begin{aligned}
L = {} & \frac{1}{2}\left(J_h + J_p + M_p\ell^2 + \frac{1}{3}\rho A\ell^3\right)\dot\theta^2(t) + \frac{1}{2}\left(M_p\Phi_{1\ell}{}^2 + \rho A\Gamma_{1_1}\right)\delta_1{}^2(t)\dot\theta^2(t) + \cdots \\
& \cdots\frac{1}{2}\left(M_p\Phi_{2\ell}{}^2 + \rho A\Gamma_{1_2}\right)\delta_2{}^2(t)\dot\theta^2(t) + \left(M_p\Phi_{1\ell}\Phi_{2\ell} + \rho A\Gamma_{1_{12}}\right)\delta_1(t)\delta_2(t)\dot\theta^2(t) + \cdots \\
& \cdots\left(\rho A\Gamma_{6_1} + M_p\ell\Phi_{1\ell} + J_p\Phi'_{1\ell}\right)\dot\theta(t)\dot\delta_1(t) + \left(\rho A\Gamma_{6_2} + M_p\ell\Phi_{2\ell} + J_p\Phi'_{2\ell}\right)\dot\theta(t)\dot\delta_2(t) + \cdots \\
& \cdots\frac{1}{2}\left(\rho A\Gamma_{1_1} + M_p\Phi_{1\ell}{}^2 + J_p\Phi'_{1\ell}{}^2\right)\dot\delta_1{}^2(t) + \frac{1}{2}\left(\rho A\Gamma_{1_2} + M_p\Phi_{2\ell}{}^2 + J_p\Phi'_{2\ell}{}^2\right)\dot\delta_2{}^2(t) + \cdots \\
& \cdots\left(\rho A\Gamma_{1_{12}} + M_p\Phi_{1\ell}\Phi_{2\ell} + J_p\Phi'_{1\ell}\Phi'_{2\ell}\right)\dot\delta_1(t)\dot\delta_2(t) + \cdots \\
& \cdots\left[KAG\Gamma_{7_1} - \frac{1}{2}KAG\left(\Gamma_{3_1} + \Gamma_{2_1}\right) - \frac{1}{2}EI\Gamma_{4_1}\right]\delta_1{}^2(t) + \cdots \\
& \cdots\left[KAG\Gamma_{7_2} - \frac{1}{2}KAG\left(\Gamma_{3_2} + \Gamma_{2_2}\right) - \frac{1}{2}EI\Gamma_{4_2}\right]\delta_2{}^2(t) + \cdots \\
& \cdots\left[KAG\Gamma_{7_{12}} + KAG\Gamma_{7_{21}} - KAG\left(\Gamma_{3_{12}} + \Gamma_{2_{12}}\right) - EI\Gamma_{4_{12}}\right]\delta_1(t)\delta_2(t)
\end{aligned} \tag{87}$$

$$D = \frac{1}{2}K_D I\dot{\delta}_1^{\,2}(t)\Gamma_{5_1} + \frac{1}{2}K_D I\dot{\delta}_2^{\,2}(t)\Gamma_{5_2} + K_D I\dot{\delta}_1(t)\dot{\delta}_2(t)\Gamma_{5_{12}} \tag{88}$$

Based on the Lagrange's principle combined with the AMM, and after tedious manipulations of extremely lengthy expressions, the established dynamic equations of motion are obtained in a matrix form by:

$$\underbrace{\begin{bmatrix} B_{11}(q) & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}}_{B(q)} \begin{bmatrix} \ddot{\theta}(t) \\ \ddot{\delta}_1(t) \\ \ddot{\delta}_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & H_{22} & H_{23} \\ 0 & H_{32} & H_{33} \end{bmatrix}}_{H} \begin{bmatrix} \dot{\theta}(t) \\ \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}}_{N(q,\dot{q})} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{22} & K_{23} \\ 0 & K_{32} & K_{33} \end{bmatrix}}_{K} \begin{bmatrix} \theta(t) \\ \delta_1(t) \\ \delta_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \tau \\ 0 \\ 0 \end{bmatrix}}_{F} \tag{89}$$

where

$$B_{11} = J_h + J_p + M_p\ell^2 + \frac{1}{3}\rho A\ell^3 + \left(M_p\Phi_{1\ell}^{\,2} + \rho A\Gamma_{1_1}\right)\delta_1^{\,2}(t) + \left(M_p\Phi_{2\ell}^{\,2} + \rho A\Gamma_{1_2}\right)\delta_2^{\,2}(t) + 2\left(M_p\Phi_{1\ell}\Phi_{2\ell} + \rho A\Gamma_{1_{12}}\right)\delta_1(t)\delta_2(t);$$

$$B_{12} = \rho A\Gamma_{6_1} + M_p\ell\Phi_{1\ell} + J_p\Phi'_{1\ell};\ B_{13} = \rho A\Gamma_{6_2} + M_p\ell\Phi_{2\ell} + J_p\Phi'_{2\ell};\ B_{21} = \rho A\Gamma_{6_1} + M_p\ell\Phi_{1\ell} + J_p\Phi'_{1\ell};$$

$$B_{22} = \rho A\Gamma_{1_1} + M_p\Phi_{1\ell}^{\,2} + J_p\Phi'^{\,2}_{1\ell};\ B_{23} = \rho A\Gamma_{1_{12}} + M_p\Phi_{1\ell}\Phi_{2\ell} + J_p\Phi'_{1\ell}\Phi'_{2\ell};$$

$$B_{31} = \rho A\Gamma_{6_2} + M_p\ell\Phi_{2\ell} + J_p\Phi'_{2\ell};\ B_{32} = \rho A\Gamma_{1_{12}} + M_p\Phi_{1\ell}\Phi_{2\ell} + J_p\Phi'_{1\ell}\Phi'_{2\ell};\ B_{33} = \rho A\Gamma_{1_2} + M_p\Phi_{2\ell}^{\,2} + J_p\Phi'^{\,2}_{2\ell};$$

$$H_{22} = K_D I\Gamma_{5_1};\ H_{23} = K_D I\Gamma_{5_{12}};\ H_{32} = K_D I\Gamma_{5_{12}};\ H_{33} = K_D I\Gamma_{5_2};$$

$$N_1 = 2\left(M_p\Phi_{1\ell}^{\,2} + \rho A\Gamma_{1_1}\right)\delta_1(t)\dot{\delta}_1(t)\dot{\theta}(t) + 2\left(M_p\Phi_{2\ell}^{\,2} + \rho A\Gamma_{1_2}\right)\delta_2(t)\dot{\delta}_2(t)\dot{\theta}(t) + 2\left(M_p\Phi_{1\ell}\Phi_{2\ell} + \rho A\Gamma_{1_{12}}\right)\left[\dot{\delta}_1(t)\delta_2(t)\dot{\theta}(t) + \delta_1(t)\dot{\delta}_2(t)\dot{\theta}(t)\right];$$

$$N_2 = -\left(M_p\Phi_{1\ell}^{\,2} + \rho A\Gamma_{1_1}\right)\delta_1(t)\dot{\theta}^2(t) - \left(M_p\Phi_{1\ell}\Phi_{2\ell} + \rho A\Gamma_{1_{12}}\right)\delta_2(t)\dot{\theta}^2(t);$$

$$N_3 = -\left(M_p\Phi_{1\ell}\Phi_{2\ell} + \rho A\Gamma_{1_{12}}\right)\delta_1(t)\dot{\theta}^2(t) - \left(M_p\Phi_{2\ell}^{\,2} + \rho A\Gamma_{1_2}\right)\delta_2(t)\dot{\theta}^2(t);$$

$$K_{22} = KAG\left(\Gamma_{3_1} + \Gamma_{2_1}\right) - 2KAG\Gamma_{7_1} + EI\Gamma_{4_1};\ K_{23} = KAG\left(\Gamma_{3_{12}} + \Gamma_{2_{12}}\right) - KAG\Gamma_{7_{12}} - KAG\Gamma_{7_{21}} + EI\Gamma_{4_{12}};$$

$$K_{32} = KAG\Gamma_{7_{12}} + KAG\Gamma_{7_{21}} - KAG\left(\Gamma_{3_{12}} + \Gamma_{2_{12}}\right) - EI\Gamma_{4_{12}};\ K_{33} = 2KAG\Gamma_{7_2} - KAG\left(\Gamma_{3_2} + \Gamma_{2_2}\right) - EI\Gamma_{4_2}$$

4. Conclusion

An investigation into the development of flexible link robot manipulators mathematical models, with a high modeling accuracy, using Timoshenko beam theory concepts has been presented.
The emphasis has been, essentially, set on obtaining accurate and complete equations of motion that display the most relevant aspects of structural properties inherent to the modeled lightweight flexible robotic structure.
In particular, two important damping mechanisms: internal structural viscoelasticity effect (Kelvin-Voigt damping) and external viscous air damping have been included in addition to the classical effects of shearing and rotational inertia of the elastic link cross-section.
To derive a closed-form finite-dimensional dynamic model for the planar lightweight robot arm, the main steps of an energetic deriving procedure based on the Lagrangian approach combined with the assumed modes method has been proposed.
An illustrative application case of the general presentation has been rigorously highlighted.
As a contribution, a new comprehensive mathematical model of a planar single link flexible manipulator considered as a shear deformable Timoshenko beam with internal structural viscoelasticity is proposed.
On the basis of the combined Lagrangian-Assumed Modes Method with specific accurate boundary conditions, the full development details leading to the establishment of a closed form dynamic model have been explicitly given.
In a coming work, a digital simulation will be performed in order to reveal the vibrational behavior of the modeled system and the relation between its dynamics and its parameters. It is also planned to do some comparative studies with other dynamic models.
The mathematical model resulting from this work could, certainly, be quite suitable for control purposes. Moreover, an extension to the multi-link case, requiring very high modeling accuracy to avoid the cumulative errors, should be a very good topic for further investigation.

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Nomenclature

A	link cross-section area
B	inertia matrix
C_D	viscoelastic material constant
D	dissipated energy
E	link Young's modulus of elasticity
F	vector of external forces
G	shear modulus
H	damping matrix
I	link moment of inertia
J_h	hub and rotor (actuator) total inertia
J_p	payload inertia
k	shear correction factor
K	stiffness matrix
K_D	Kelvin-Voigt damping coefficient
ℓ	link length
L	Lagrangian
M	bending moment
M_p	payload mass
n	mode number
N	vector of Coriolis and centrifugal forces
q	vector of generalized coordinates
S	shear force
t	time
T	kinetic energy
U	stored potential energy
$w(x,t)$	transverse deflection

x	coordinate along the beam
$\Phi_n(x)$	n th transverse mode shape
$\Psi_n(x)$	n th rotational mode shape
$a(x,t)$	angular position of a point of the deflected link
δ_n	n th modal amplitude
ε	strain
ν	normal stress
$\theta(t)$	angular position of the rotating X -axis
ρ	link uniform linear mass density
σ	shear angle
τ	actuator torque applied at the base of the link
ω_n	n th natural frequency of vibration
ω_{d_n}	n th damped natural frequency of vibration
$\gamma(x,t)$	rotation of cross-section about neutral axis
$\eta(t)$	rotating X -axis angular position
ξ_n	n th damping ratio.

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The purpose of this volume is to encourage and inspire the continual invention of robot manipulators for science and the good of humanity. The concepts of artificial intelligence combined with the engineering and technology of feedback control, have great potential for new, useful and exciting machines. The concept of eclecticism for the design, development, simulation and implementation of a real time controller for an intelligent, vision guided robots is now being explored. The dream of an eclectic perceptual, creative controller that can select its own tasks and perform autonomous operations with reliability and dependability is starting to evolve. We have not yet reached this stage but a careful study of the contents will start one on the exciting journey that could lead to many inventions and successful solutions.

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