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An Interval Metric

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1. Introduction

The measurement of distance has become very important due to significant applications in various fields such as remote sensing, data mining, pattern recognition and multivariate data analysis. Researchers in these areas often face problems that have essentially noise distance, but no real metric capture this problem model. Accordingly, we propose a distance that extends the concept of generic metric of real numbers to an interval metric, where the distance between two elements of a set can be some interval measure. We see the use of distances in several areas, such as cluster algorithms for automatic classification of data of high dimensionality in the work of Fu & Huang (2007), and in problems of segmentation for audio by Sundaram & Narayanan (2007). Both use the K-means algorithm in their work.

We have the concepts of acoustic distance and phonemic distance in the work by Lin & Lee (2007). The proposed distance notion can be used also to compare histograms used as feature indexing of images in databases in environments of content-based image retrieval, CBIR, because the traditional metric distance between two histograms of two similar images can be as large as the distance of two histograms of two very different images and that is unwanted in a CBIR system.

Girard & Pappas (2007) propose the use of metric for approximating discrete and continuous systems, a comparison of languages accepted by automata. Many works in the area of fuzzy numbers deal or propose metrics, such as Fono et al. (2007) which use metric in Fuzzy sets, but it does not preserve uncertainties. Balopoulos et al. (2007) suggest a family of distances and similarity fuzzy measures based on the matrice norm. In the image processing area we have several works by Yu et al. (2006) which propose a robust method for distances metrics for estimation of similarity. Manouvrier et al. (2005) propose a distance between images recursively partitioned into structures of trees, which is effective in CBIR techniques. Georgiou et al. (2007) propose a metric distance between distributions in the image segmentation process.

One of the pioneers in the fuzzy distance approach that preserves uncertainty was Voxman (1998) which addresses principles of distance on the fuzzy point of view and treats the principle of convergence on the view of the Cauchy sequences. He was the first to propose a fuzzy distance between fuzzy numbers.

In this work we propose an extension of the concept of real metric for a concept of interval metric. In this sense the distance that we propose is a generalisation of the Euclidean distance. In our approach the distance between two intervals is an interval, without losing the characteristics of the Euclidean metric when it leads with real numbers or degenerated intervals. The metric proposed here, beyond to providing the needs of the areas mentioned above, it preserves the monotonic inclusion of the Moore arithmetic Moore (1966), since this does not include the main feature of their arithmetic which is the property of monotonic inclusion, beyond it is not strictly interval because the distance between two intervals of uncertainty is a real number.

Of all works cited here that more close to our idea is the work by Chakraborty & Chakraborty (2006) which proposes a fuzzy distance for fuzzy numbers, which preserves a distance uncertainties. The authors put a natural question: " if we do not know the numbers exactly how can the distance between them be an exact value? ". At the same time, they criticise the use of the supreme, of the minimal or of any other candidate as absolute representative of the distance between two fuzzy numbers. They also consider the distance between two fuzzy numbers as a fuzzy number, saying that the distance between two numbers with uncertainties must be a number with uncertainty. In addition, for fuzzy sets, Grzegorzewski (2004) uses a Hausdorff metric in the construction of a fuzzy metric, that unfortunately it does not preserve uncertainty. In their works they use metric spaces and topological spaces.

The proposed metric beyond of being a generalisation of the Euclidean metric in the reals it includes both the logic part of fuzzy logic as well the numerical part. This metric opens many possibilities for research and it possible to say that it creates a new paradigm of metrics. Much of mathematics is based on the notion of distance. This notion is fundamental to building the principles of calculus such as limit and continuity. As mathematical is a tool for many areas of knowledge, the importance of distance metric impact several areas such as signal processing, robust control systems, neural network that deal directly or indirectly with this notion. Often it represents characteristics of systems objects with uncertainties, where these uncertainties can be generated by the following factors: lack of precision of sensors, precision in mathematical representation of the system, the limitation of implementation in machine arithmetic. Therefore these areas need a model of distance that captures the uncertainties inherent in their processes. Algorithms for classification of patterns such as K-means, the SOM, support vector machine (SVM), algorithms for retrieval (CBIR), genetic algorithms and others using the very notion of distance in the separation of inaccurate data noisy. Often these area researchers are faced with problems that is essentially a distance noisy, but no real metric captures this type of problem.

According we propose an extension of the real metric to an "interval metric ", where a distance between two intervals is an interval, without losing the characteristics of the Euclidean metric when it manipulates real numbers or degenerated intervals. We see the need for this metric because the one proposed by Moore Moore (1966) does not include the main feature of the arithmetic that is the property of inclusion monotony, and it is not strictly interval, because the distance between two intervals of uncertainty is a real number. We propose this metric in order to increase the power of representation of interval mathematics. With this metric it is possible to formulate new concepts of interval sequences, interval limits, thereby reshaping

the concepts of interval integral, derivative, complex variables, analysis of convergence and stability of LTI (Linear Time Invariant) systems. We present the definition of distance and we built versions of some results with this distance. In this work the distance has the role of supporting the definition of a module that preserves uncertainty for application in the convergence analysis of LTI systems.

2. Interval Mathematics

Mathematics has been successful as a language in knowledge construction. It allows us to create language from abstractions of real entities. Like any language it has its limitations, one of which is not to have an algorithmic representation for real numbers. This representation has been a topic of research since Pythagoras to nowadays. In the 50's Sunaga and Moore proposed an interval to control errors for handling real numbers. In their work they describe the interval arithmetic that is, in some way, an extension of real arithmetic. One use this approach to develop several branches applications such as linear systems to model real systems for digital signal processing. Here, we use the interval set $\mathbb{IR}$ as discussed by Vaccaro (2001). Accuracy of mathematical calculations in various areas of science and technology has been the subject of scientific work, always seeking the development of arithmetic algorithms, aiming to achieve the best possible accuracy in the processing of numerical data as seen in Marciniak (2003) and Popova (1994). This is not always possible due some factors such as: lack of precision of input data, imprecision of floating-point arithmetic and the physical limitations of the machines. As mentioned earlier, we can see that it is not a simple problem and that it covers the entire computer system, including its logical representation, mathematical modelling, memory capacity, size of words, floating-point arithmetic and so on. We focus mainly on mathematical representation by interval approach, because here the arithmetic operations on real numbers is invariant by interval arithmetics. The pioneers works that marked the beginning of the development of interval arithmetic was Moore (1966) and Sunaga (1958). This research area of mathematics is mainly interested in solving the mathematical expressions that can be performed by computers. Therefore, it is crucial that this approach responds to the questions of accuracy and efficiency which arises in the practice of scientific computing. Despite the success of interval mathematics in the field of computing science, the interval analysis has been not a similar success as, for exemplo, the theory of complex variables as an extension of real analysis. More seriously was the fact that the interval analysis was not successful as a basis for interval computation. Perhaps this was mainly because the researchers in this area have insisted on a metric which does not capture the interval approach. Nowadays, one has observed that the classical binary logic is not so much adequate for the conversion of the real world to a virtual world of representation that makes the fuzzy logic a so good alternative to account this problem. The operations of fuzzy logic in the interval $[0,1]$ can be bijectively mapped with the interval $-\infty, +\infty$, on which one can work with the operations of interval arithmetic. In this way we can see that the interval arithmetic is more suited than the fuzzy logic to approach the traditional real arithmetic. We can also imagine the same situation when the range $[0,1]$ represents probability space, which is also home to a well interval arithmetic.

Definition 1 (Interval Representation). *A function $F : \mathbb{IR}^m \rightarrow \mathbb{IR}^n$ is called an **interval representation** of a real function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ if, for each $\vec{X} \in \mathbb{IR}^m$ and $\vec{x} \in \vec{X}$, $f(\vec{x}) \in F(\vec{X})$.*

Definition 2 (Interval). Let a and $b \in \mathbb{R}$ be such that $a \leq b$. The set $X = \{x \in \mathbb{R} : a \leq x \leq b\}$ is called an **interval** and will be denoted by $X = [a; b]$. The set $X = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}$. The set of all intervals will be represented by $\mathbb{IR}$.

Each interval has associated to it two projections, π_1 and π_2 , defined by $\pi_1([a; b]) = a$ and $\pi_2([a; b]) = b$. In order to simplifying notation we will use $\underline{X}$ for representing $\pi_1(X)$ and $\overline{X}$ for $\pi_2(X)$. Let $F : \mathbb{IR} \rightarrow \mathbb{IR}$ be an interval function. The **lower limit** of $F(X)$ is the semi-interval function $\underline{F}(X) : \mathbb{IR} \rightarrow \mathbb{R}$, where $\underline{F}(X) = \pi_1(F(X))$ and the **upper limit** of $F(X)$ is the semi-interval function $\overline{F}(X) : \mathbb{IR} \rightarrow \mathbb{R}$, where $\overline{F}(X) = \pi_2(F(X))$.

Definition 3 (Diameter of Interval). Let $X \in \mathbb{IR}$ be an interval. The **diameter** of the interval X is defined by the non-negative real number $\text{Diam}(X) = \overline{X} - \underline{X}$.

The radius of an interval, X , is defined as the half diameter of X , as it is shown in the following equation:

$$\text{raio}(X) = \frac{\text{Diam}(X)}{2}. \quad (1)$$

Definition 4 (Inclusion Order). Let X and $Y \in \mathbb{IR}$. We say that $X \subseteq Y$ if only if $\underline{Y} \leq \underline{X}$ and $\overline{X} \leq \overline{Y}$.

Definition 5 (Kulisch-Miranker Order). Let X and $Y \in \mathbb{IR}$. X is **least or equal** to Y , denoted by $X \leq Y$, if $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.

If $X \leq Y$ and $X \cap Y = \emptyset$. Then we say that $X \prec Y$, which is equivalent to say that $Y \succ X$. An interval, X , is said to be **positive** if $\underline{X} > 0$ and **negative** if $\overline{X} < 0$.

Definition 6 (Middle Point of an Interval). Given $X \in \mathbb{IR}$ we define the **middle point** of X as the real number given by

$$\text{pm}(X) = \frac{\underline{X} + \overline{X}}{2}.$$

Definition 7 (The greatest lower bound of set of intervals or infimum). Let $M \subseteq \mathbb{IR}$. The **greatest lower bound** or **infimum** of M , with regard to the order $\leq$, denoted by $\text{Inf}m(M)$, is the interval Y such that $Y \leq X$, $\forall X \in M$ and given any interval $Z \leq X \forall X \in M$, then $Z \leq Y$.

Proposition 1. Let $M \subseteq \mathbb{IR}$. The infimum of M , with regard the order $\leq$, is given by the equation (2).

$$\text{Inf}m(M) = [\inf\{\underline{X} : X \in M\}; \inf\{\overline{X} : X \in M\}]. \quad (2)$$

Proof. By the definition 7 we have that $\text{Inf}m(M) \leq X$, $\forall X \in M$. Thus by definition 5 $\text{Inf}m(M) \leq \underline{X}$, $\forall X \in M$ and $\overline{\text{Inf}m(M)} \leq \overline{X}$, $\forall X \in M$. Therefore $\text{Inf}m(M) = [\inf\{\underline{X} : X \in M\}; \inf\{\overline{X} : X \in M\}]$. ■

The infimum is greatest of the lesser bounds of the set and when it is applied to a degenerate interval set it coincides with the notions of infimum of the real number set.

Definition 8 (Minimum of an Interval). Let $M \subseteq \mathbb{IR}$ and $X \in \mathbb{IR}$, we say that X is the **minimum** of M , with regard to the order $\leq$, denoted by $\text{Min}(M)$, if X is the infimum of M , with regard to the order $\leq$ and $X \in M$.

3. Semantics of Intervals

There are several semantics to $X \subseteq \mathbb{IR}$. Here we emphasize only two of them, that which treats the interval as a real envelopment and that which sees the interval as a numeric entity. Each one of them has its advantages and disadvantages.

3.1 Intervals as Envelopment of Reals

Researchers who take this semantic for the set $\mathbb{IR}$ see each interval as a wrapper that has information of a real number. From this point of view the multiplication of an interval $X \in \mathbb{IR}$ by itself not always has the same result that the multiplication proposed by Moore. In this approach X^2 is always a nonnegative interval, since it represents the same real number. This interpretation is usually accepted in the context of interval arithmetic. According to this semantics an interval of real numbers is a real subject to uncertainties, ie

$$\forall x \in \mathbb{R}, X \text{ represents } x \leftrightarrow x \in X.$$

Thus, any real number belonging to the envelopment is a possible representative of the real number which the envelopment represents. The effect of envelopment is modelling the propagation of error in numerical calculation in floating point Vaccaro (2001).

3.2 Intervals as Interval-numbers

In this approach an interval is seen as a mathematical entity that represents the real numbers and intervals belonging in it. It constitutes a different type of information from that it conveys, it is a new type of number whose semantics can be defined as follows:

$$\forall X \in \mathbb{IR}, Y \text{ represents } X \leftrightarrow X \subseteq Y.$$

In words, an interval represents all the real intervals it contains, in particular, the real numbers regarded as degenerate intervals.

Thus, an interval-number represents all the subsets and not only the real numbers individually selected within a field of uncertainty. The concept of interval exceeds the trichotomic law of the real numbers because intervals can contain numbers both positive and negative.

The basic concepts and notations presented here can be found in Oliveira et al. (1997), Vaccaro (2001), Trindade (2002), Santiago et al. (2006)

3.2.1 Moore Interval Arithmetic

Definition 9 (Arithmetic Operations in $\mathbb{IR}$). *Let $X, Y \in \mathbb{IR}$ be two real intervals. The operations of addition, subtraction, multiplication and division in $\mathbb{IR}$ are defined by $X * Y = \{x * y : x \in X, y \in Y\}$, where $*$ $\in \{+, -, \times, \div\}$ is one of the four arithmetic operations. If ω is an unary arithmetic operation, then $\omega(X)$ is defined by $\omega(X) = \{\omega(x) : x \in X\}$.*

Note: It should be noted that for the operation of division, it is necessary to assume that $0 \notin Y$, because, otherwise, the operation is not well defined.

The following proposition will be not proved, because it is a basic concept already available in the literature, but its proof can be found eg in Oliveira et al. (1997) in the form of several theorems.

Proposition 2. *If $X, Y \in \mathbb{IR}$ are two real intervals, then*

1. **Interval Addition** $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$.

2. **Interval Pseudo Inverse Additive** $-X = [-\overline{X} - \underline{X}]$.
3. **Interval Subtraction** $X - Y = X + (-Y) = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}]$.
4. **Interval Multiplication** $= X \cdot Y = [\min\{\underline{X} \cdot \underline{Y}, \underline{X} \cdot \overline{Y}, \overline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}\}; \max\{\underline{X} \cdot \underline{Y}, \underline{X} \cdot \overline{Y}, \overline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}\}]$.
5. **Interval Pseudo inverse Multiplicative** $X^{-1} = 1/X = [\frac{1}{\overline{X}}, \frac{1}{\underline{X}}]$ and $0 \notin X$.
6. **Interval Division** $\frac{X}{Y} = X \cdot Y^{-1} = [\min\{\frac{\underline{X}}{\underline{Y}}, \frac{\underline{X}}{\overline{Y}}, \frac{\overline{X}}{\underline{Y}}, \frac{\overline{X}}{\overline{Y}}\}; \max\{\frac{\underline{X}}{\underline{Y}}, \frac{\underline{X}}{\overline{Y}}, \frac{\overline{X}}{\underline{Y}}, \frac{\overline{X}}{\overline{Y}}\}]$ with $0 \notin [\underline{Y}, \overline{Y}]$.

Proposition 3. If X, Y and $Z \in \mathbb{IR}$ are real intervals, then

1. **Algebraic Properties of Addition in $\mathbb{IR}$**

- **Closeness:** If $X \in \mathbb{IR}$ and $Y \in \mathbb{IR}$ then $X + Y \in \mathbb{IR}$;
- **Associativity:** $(X + Y) + Z = X + (Y + Z)$;
- **Commutativity:** $X + Y = Y + X$;
- **Neutral Element:** $\exists! \mathbf{0} = [0; 0] \in \mathbb{IR}$ such that: $X + \mathbf{0} = \mathbf{0} + X = X$

Proposition 4. If X, Y and $Z \in \mathbb{IR}$ are real intervals, then

1. **Algebraic Properties of the Multiplication in $\mathbb{IR}$**

- **Closeness:** If $X \in \mathbb{IR}$ and $Y \in \mathbb{IR}$, then $X \cdot Y \in \mathbb{IR}$;
- **Associativity:** $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$;
- **Commutativity:** $X \cdot Y = Y \cdot X$;
- **Neutral Element:** $\exists! \mathbf{1} = [1; 1] \in \mathbb{IR}$ such that: $X \cdot \mathbf{1} = \mathbf{1} \cdot X = X$;
- **Subdistributivity:** $X \cdot Y + Z \subseteq X \cdot Y + X \cdot Z$;

2. **Consequence of the Pseudo Inverse Multiplicative:** Let X be an interval such that $0 \notin X$. Then $1 \in X/X$

One of the main properties of the Moore arithmetic is the monotonic property. It guarantees the interval correctness and the error inclusion.

Definition 10 (Interval Canonical Representation - CIR). Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a non-asymptotic function ¹ The function $CIR(f) : \mathbb{IR}^m \rightarrow \mathbb{IR}^n$ is a **canonical representation** of the function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ if CIR is defined by

$$CIR(f)(\vec{X}) = [\inf\{f(\vec{x}) : \vec{x} \in \vec{X}\}; \sup\{f(\vec{x}) : \vec{x} \in \vec{X}\}].$$

In other words, $CIR(f)$ is an interval function that maps an m-tuple $\vec{X}$ in the lower n-tuple of intervals that contain $f(\vec{X})$.

Proposition 5. Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be not asymptotic. Then $\forall \vec{X} \in \mathbb{IR}^m$ and $\vec{x} \in \vec{X}$ we have $f(\vec{x}) \in CIR(f)(\vec{X})$.

Proof. If $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is a total function and it is not asymptotic and $\vec{x} \in \vec{X}$, then $\inf\{f(\vec{a}) : \vec{a} \in \vec{X}\} \leq \vec{x} \leq \sup\{f(\vec{a}) : \vec{a} \in \vec{X}\}$. Therefore, by definition 10, we have $f(\vec{x}) \in CIR(f)(\vec{X})$. ■

¹ A real function f is called asymptotic if for a given interval X the set $\{f(a) : \underline{X} < a < \overline{X}\}$ not have the smallest element or not have the largest element.

4. Topology of $\mathbb{IR}$

In this section we will present some topological properties of the space $\mathbb{IR}$ as a metric space. The properties presented here are based on the notion of proximity and limit as it is the case of distance. Because only it will be shown some topological properties of $\mathbb{IR}$ the proof will be omitted. The objective here is to compare properties in the following sections. Anyway the interested readers can found the proofs in Moore (1966), Moore (1979), Oliveira et al. (1997) and Trindade (2002).

4.1 Basic Definitions

In the following will be presented some definitions on the topology of $\mathbb{IR}$.

4.2 Distance

A function $d_e : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$, defined by $d_e(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$, is called **Euclidian distance** from $\vec{x}$ to $\vec{y}$ in $\mathbb{R}^m$. It has the following properties:

- (D₁) $d_e(\vec{x}, \vec{y}) \geq 0$ and $d_e(\vec{x}, \vec{y}) = 0 \Leftrightarrow \vec{x} = \vec{y}$ (positive definite);
- (D₂) $d_e(\vec{x}, \vec{y}) = d_e(\vec{y}, \vec{x})$ (symmetrical);
- (D₃) $d_e(\vec{x}, \vec{y}) \leq d_e(\vec{x}, \vec{z}) + d_e(\vec{z}, \vec{y})$ (triangular inequality).

Definition 11. Let A be any set. A function $d : A \times A \rightarrow \mathbb{R}$ is called a **metric** on A if it satisfies the following properties:

1. reflexivity: $d(x, x) = 0$;
2. triangular inequality: $d(x, z) \leq d(x, y) + d(y, z)$;
3. symmetry: $d(x, y) = d(y, x)$;
4. indiscernible identity: if $d(x, y) = 0$ then $x = y$.

Definition 12. A distance $d : A \times A \rightarrow \mathbb{R}$ is called **quasi-metric** if it satisfies the following properties:

1. reflexivity: $d(x, x) = 0$;
2. triangular inequality: $d(x, z) \leq d(x, y) + d(y, z)$;
3. indiscernible symmetrical identity: if $d(x, y) = d(y, x) = 0$, then $x = y$.

Definition 13 (Moore Distance). The **Moore distance**, $D_M : \mathbb{IR}^2 \rightarrow \mathbb{R}$, between X and $Y \in \mathbb{IR}$, is given by

$$D_M(X, Y) = \max\{|\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}|\}.$$

Theorem 1. The Moore distance, above defined, is a metric on $\mathbb{IR}$.

Proof. See Moore (1962). ■

Geometrically, a distance between two interval is the length of the largest segment that separates the extremes of the intervals.

Definition 14 (Moore Interval Module Moore (1979)). Let $X \in \mathbb{IR}$ be an interval. The **module or norm** of the interval X is defined as the non-negative real number, $|X|_M = D_M(X, \mathbf{0})$, which corresponds to the distance from X to zero.

In other words, $|X|_M = D_M(X, \mathbf{0}) = \max\{|\underline{X}|, |\overline{X}|\} \geq 0$.

Theorem 2 (Moore Interval Module properties).

1. $|X|_M = 0 \Leftrightarrow X = 0$;
2. $|X + Y|_M \leq |X|_M + |Y|_M$;
3. $|X \cdot Y|_M = |X|_M \cdot |Y|_M$.

Proof. It follows immediately from the module definition. ■

The figure 1 gives a geometric interpretation to the module of an interval.

Fig. 1. Geometric interpretation to the module of an interval

Geometrically, the module of an interval is the length of the largest segment which joins the extremes of the interval to the origin.

Theorem 3. Let $X, Y, Z, W \in \mathbb{IR}$ be intervals. Then, the following properties are true.

1. $D_M(X + Y, X + Z) = D_M(Y, Z)$;
2. $D_M(X \cdot Y, X \cdot Z) \leq |X|_M \cdot D_M(Y, Z)$;
3. $D_M(X + Y, Z + W) \leq D_M(X, Z) + D_M(Y, W)$;
4. $D_M(X \cdot Y, Z \cdot W) \leq |Y|_M \cdot D_M(X, Z) + |Z|_M \cdot D_M(Y, W)$;
5. $D_M(X, Y) \leq |X|_M \cdot |Y|_M \cdot D_M(\frac{1}{X}, \frac{1}{Y})$, if $0 \notin X, 0 \notin Y$;
6. $X \subseteq Y \Rightarrow |X|_M \leq |Y|_M$;

Proof. See Oliveira et al. (1997). ■

Geometrically, the diameter of an interval is the length of the segment which joins the extremes of the interval.

Definition 15. Let M be any set. A function $d : M \times M \rightarrow \mathbb{IR}$, is said an **interval metric** if it satisfies the following properties:

1. *reflexivity:* $0 \in d(X, X)$,
2. *triangular inequality* $|d(X, Y)|_M \leq |d(X, Z)|_M + |d(Z, Y)|_M$
3. *symmetry:* $d(X, Y) = d(Y, X)$
4. *indiscernible identity:* if $0 \in d(X, Y) = d(X, X) = d(Y, Y)$ then $X = Y$.

Definition 16. [An Interval Distance] Let X and $Y \in \mathbb{IR}$. An Interval distance between X and Y , denoted by $m_{ei}(X, Y)$, is defined by

$$m_{ei}(X, Y) = [\inf\{d_e(x, y) : x \in X \text{ and } y \in Y\}, \sup\{d_e(x, y) : x \in X \text{ and } y \in Y\}].$$

Proposition 6. Let X and Y be two intervals, with $X \leq Y$ and $X \cap Y = \emptyset$. Then

$$m_{ei}(X, Y) = [\underline{Y} - \overline{X}, \overline{Y} - \underline{X}].$$

Proof. If $x \in X$ and $y \in Y$, then $x \leq \overline{X}$ and $y \geq \underline{Y}$. As $X \leq Y$ and $X \cap Y = \emptyset$, then $|x - y| = y - x$. Therefore, $y - x \geq \underline{Y} - \overline{X}$ and, so, $\underline{Y} - \overline{X} = \min\{|x - y| : x \in X \text{ and } y \in Y\}$. Similarly, it is possible to prove that $\overline{Y} - \underline{X} = \max\{|x - y| : x \in X \text{ and } y \in Y\}$. ■

Proposition 7. Let X and Y be two interval, with $X \leq Y$ and $X \cap Y \neq \emptyset$. Then

$$m_{ei}(X, Y) = [0, \overline{Y} - \underline{X}].$$

Proof. As $X \cap Y \neq \emptyset$, $\exists z \in X \cap Y$. Therefore, $0 \in \{d_e(x, y) : x \in X \text{ and } y \in Y\}$. Then, $\inf\{d_e(x, y) : x \in X \text{ and } y \in Y\} = 0$. Because $X \leq Y$, $\underline{X}$ is the least element of X and $\overline{Y}$ is the greatest element of Y , it follows that $\overline{Y} - \underline{X} = |\overline{Y} - \underline{X}| = \max\{d_e(x, y) : x \in X \text{ and } y \in Y\}$. Therefore, $m_{ei}(X, Y) = [0, \overline{Y} - \underline{X}]$. ■

Proposition 8. Let X and Y be two intervals, with $X \subseteq Y$. Then

$$m_{ei}(X, Y) = [0, \max\{\overline{X} - \underline{Y}, \overline{Y} - \underline{X}\}].$$

Proof. As $X \subseteq Y$, $X \cap Y \neq \emptyset$ and $\min\{d_e(x, y) : x \in X \text{ and } y \in Y\} = 0$. If $X \subseteq Y$, then $\underline{Y} \leq \underline{X} \leq \overline{X} \leq \overline{Y}$. Let $x \in X$ and $y \in Y$. If $y \leq \overline{X}$, then $|x - y| \leq \overline{X} - \underline{Y}$. If $y > \overline{X}$, then $|x - y| = y - x \leq \overline{Y} - \underline{X}$. So $|x - y| \leq \max\{\overline{X} - \underline{Y}, \overline{Y} - \underline{X}\}$. Therefore, $m_{ei}(X, Y) = [0, \max\{\overline{X} - \underline{Y}, \overline{Y} - \underline{X}\}]$. ■

Corollary 1. Let $X \cap Y \neq \emptyset$. Then $m_{ei}(X, Y) = [0, \max\{\overline{X} - \underline{Y}, \overline{Y} - \underline{X}\}]$.

Proof. It is a direct application of the propositions 7 and 8. ■

Proposition 9. The distance m_{ei} coincides with the Euclidian distance, d_e , when it is applied to degenerate intervals. So, if $X = [x, x]$ and $Y = [y, y]$, then

$$m_{ei}(X, Y) = [d_e(x, y), d_e(x, y)].$$

Proof. As $X = [x, x]$ and $Y = [y, y]$, it follows that

$$\begin{aligned} m_{ei}(X, Y) &= [\min\{d_e(x, y) : x \in X \text{ and } y \in Y\}, \max\{d_e(x, y) : x \in X \text{ and } y \in Y\}] \\ &= [\min\{d_e(x, y) : x \in \{x\} \text{ and } y \in \{y\}\}, \max\{d_e(x, y) : x \in \{x\} \text{ and } y \in \{y\}\}] \\ &= [\min\{d_e(x, y)\}, \max\{d_e(x, y)\}] \\ &= [d_e(x, y), d_e(x, y)]. \end{aligned}$$

Corollary 2. The distance m_{ei} , restricted to degenerate intervals, is an interval metric.

Proof. It follows directly from proposition 9. ■

Corollary 3. If $X \cap Y \neq \emptyset$. Then $m_{ei}(X, Y) = [0, \max\{\overline{X} - \underline{Y}, \overline{Y} - \underline{X}\}]$.

Proof. It follows directly from propositions 7 and 8. ■

Proposition 10. The distance m_{ei} is the CIR of the Euclidian distance.

Proof. It is a direct consequence of the definitions 10 and 16. ■

Theorem 4 (CIR of a metric is an interval metric). *Let A be a set and d a metric. Then $CIR(d)$ is an interval metric.*

Proof. $CIR(d)(X, Y)$ satisfies the four proprieties of interval metric.

- **Reflexivity:** $0 \in CIR(d)(X, X)$, because $0 = d(x, x) \in \{d(x, y) : x \text{ and } y \in X\}$.
- **Triangular inequality:** $|CIR(d)(X, Y)|_M \leq |CIR(d)(X, Z)|_M + |CIR(d)(Z, Y)|_M$, since $\max\{d(x, y) : x \in X \text{ and } y \in Y\} \leq \max\{d(x, z) : x \in X \text{ and } z \in Z\} + \max\{d(z, y) : z \in Z \text{ and } y \in Y\}$
- **Symmetry:** $CIR(d)(X, Y) = CIR(d)(Y, X)$, because $\min\{d(x, y) : x \in X \text{ and } y \in Y\} = \min\{d(y, x) : x \in X \text{ and } y \in Y\}$ and $\max\{d(x, y) : x \in X \text{ and } y \in Y\} = \max\{d(y, x) : x \in X \text{ and } y \in Y\}$.
- **indiscernible Identity:** We intend to prove that if $0 \in CIR(d)(X, Y) = CIR(d)(X, X) = CIR(d)(Y, Y)$, then $X = Y$. For that, suppose that $X \neq Y$. If $X \cap Y = \emptyset$, then $0 \notin CIR(d)(X, Y)$ absurd, by the hypothesis. Case $X \cap Y \neq \emptyset$ we have four possible cases. Case $X \subset Y$ or $Y \subset X$ $CIR(d)(X, X) \neq CIR(d)(Y, Y)$ also it absurd from the hypothesis. The others two cases $X \leq Y$ or $Y \leq X$ both cases or $CIR(d)(X, Y) \neq CIR(d)(X, X)$ or $CIR(d)(X, Y) = CIR(d)(Y, Y)$ which also is absurd by hypothesis. So, $CIR(d)$ satisfies the indiscernible property. ■

We believe that in the extension of real representation to interval representation there is a shift of paradigm, by which we mean that the most theorems valid for real numbers need to be adapted for interval version.

The same we can say about the properties of definition 11, because if we do an analysis of the reflexive property we observe that it is good for real numbers since they are non dimensional when they are seen as point of real line, by the impossibility of physical existence. In this case the distance from it to itself is zero. However if the entity is an extensive body it reasonable that a least distance from it to itself be zero and the largest distance be the measure of the extension of its body. We observe that in this case the distance would be an interval whose extremes is zero and the measure of the extension of the body. So, we can see that the distance from an interval to itself may zero when one measures from a point of the interval to itself or its diameter when one measures the distance between its extremes.

We can observe one more inconsistency in the use of a propriety of a set whose elements are dimensionless when extended to those with extensive features, as the Mooore distance, whose distance is a real number. In a semantic field where the interval are used to represent uncertainties it reasonable to expect that given two intervals X and Y the distance between them be an interval of uncertainties that varies between $\min\{d_e(x, y) : x \in X \text{ and } y \in Y\}$ and $\max\{d_e(x, y) : x \in X \text{ and } y \in Y\}$.

Proposition 11. *The distance m_{ei} is an interval metric.*

Proof. It is a direct consequence of proposition 10 and theorem 4. ■

Proposition 12. *Let X and $Y \in \mathbb{IR}$, $m_{ei}(X, Y) \leq [0, Diam(Y)]$ if only if $X \subseteq Y$.*

Proof. $\Leftarrow$) Suppose that $X \subseteq Y$, then it satisfies the definition 4 and the proposition 8, so we have

$$m_{ei}(X, Y) = [0, \max\{\bar{X} - \underline{Y}, \bar{Y} - \underline{X}\}]. \quad (3)$$

By the hypothesis that $X \subseteq Y$ and by the definition 4 we conclude that

$$\max\{\bar{X} - \underline{Y}, \bar{Y} - \underline{X}\} \leq \bar{Y} - \underline{Y} = Diam(Y). \quad (4)$$

By the equation (4) and the definition 5 we have

$$m_{ei}(X, Y) \leq [0, Diam(Y)]. \quad (5)$$

$\Rightarrow$) Now, suppose that $m_{ei}(X, Y) \leq [0, Diam(Y)]$ is within the three possible cases and its symmetric $m_{ei}(X, Y)$ satisfies at least one of the propositions 6, 7 ou 8. Assuming that $m_{ei}(X, Y)$ satisfies the conditions of proposition 6 we have

$$m_{ei}(X, Y) = [\underline{Y} - \bar{X}, \bar{Y} - \underline{X}]. \quad (6)$$

By the hypothesis that $m_{ei}(X, Y)$ satisfies the conditions of the proposition 6 and by the equation (6) we have

$$\begin{aligned} \underline{Y} - \bar{X} &\geq 0 \\ \bar{Y} - \underline{X} &\geq \bar{Y} - \underline{Y}. \end{aligned} \quad (7)$$

By the equation (7) and the definition 5 we conclude that

$$[0, Diam(Y)] \leq m_{ei}(X, Y). \quad (8)$$

By the equation (8) we conclude if $m_{ei}(X, Y)$ satisfies the conditions of the proposition 6, then it does not satisfies the conditions of the proposition 12.

Suppose that $m_{ei}(X, Y)$ satisfies the conditions of proposition 7 then

$$m_{ei}(X, Y) = [0, \bar{Y} - \underline{X}]. \quad (9)$$

By the hypothesis that $m_{ei}(X, Y)$ satisfies the conditions of the propositions 7 and by the equation (9) we have

$$\begin{aligned} 0 &\geq 0 \\ \text{and} \\ (\bar{Y} - \underline{X}) &\geq (\bar{Y} - \underline{Y}). \end{aligned} \quad (10)$$

By the equation (10) and the definition 5 we conclude that

$$\begin{aligned} m_{ei}(X, Y) &\leq [0, Diam(Y)] \\ \text{only if} \\ \underline{X} &= \underline{Y}. \end{aligned} \quad (11)$$

By the equation (11) we conclude that if $m_{ei}(X, Y)$ satisfies the conditions of proposition 7, it satisfies also the conditions of proposition 12 only if the equality $\underline{X} = \underline{Y}$ is satisfied. Thus $X \subseteq Y$.

Suppose, now, that $m_{ei}(X, Y)$ satisfies the conditions of the proposition 8, then we get

$$m_{ei}(X, Y) = [0, \max\{\bar{X} - \underline{Y}, \bar{Y} - \underline{X}\}]. \quad (12)$$

By the hypothesis that $m_{ei}(X, Y)$ satisfies the condition 8 and by the equation (12) we have

$$\begin{aligned} 0 &\leq 0 \\ \text{and} \\ (\bar{Y} - \underline{Y}) &\leq \max\{(\bar{X} - \underline{Y}), (\bar{Y} - \underline{X})\}. \end{aligned} \quad (13)$$

By the equation (13) and the definition 4 we conclude that

$$\begin{aligned} m_{ei}(X, Y) &\leq [0, \text{Diam}(Y)] \\ \text{and} \\ X &\subseteq Y. \end{aligned} \quad (14)$$

Finally, by the equations (8), (11) and (14), we conclude that if $m_{ei}(X, Y) \leq [0, \text{Diam}(Y)]$, then $X \subseteq Y$. ■

Proposition 13. Let X and $Y \in \mathbb{IR}$ be such that $X \neq Y$. Then, $m_{ei}(X, Y) \leq [0, \text{Diam}(Y) + \text{Diam}(X)]$ if only if $X \cap Y \neq \emptyset$.

Proof. The case where $X \subseteq Y$ was proved in the proposition 12. For the other cases we analyse the three possible cases for $m_{ei}(X, Y)$, which are the cases of the propositions 6, 7 and 8. Suppose that $m_{ei}(X, Y)$ satisfies the conditions of the proposition 6, then we get

$$m_{ei}(X, Y) = [\underline{Y} - \bar{X}, \bar{Y} - \underline{X}]. \quad (15)$$

By the hypothesis that $m_{ei}(X, Y)$ satisfies the conditions of the proposition 6 and the equation (15) we conclude

$$\begin{aligned} [0, \bar{Y} - \underline{Y} + \bar{X} - \underline{X}] &\leq [\underline{Y} - \bar{X}, \bar{Y} - \underline{X}] \\ [0, \text{Diam}(Y) + \text{Diam}(X)] &\leq m_{ei}(X, Y). \end{aligned} \quad (16)$$

By the equation (16), we conclude that if $m_{ei}(X, Y)$ satisfies the conditions of the proposition 6 it does not satisfies the conditions of proposition 13.

Suppose that $m_{ei}(X, Y)$ satisfies the conditions of proposition 7 then we get

$$m_{ei}(X, Y) = [0, \bar{Y} - \underline{X}]. \quad (17)$$

By the hypothesis that $m_{ei}(X, Y)$ satisfies the conditions of the proposition 7 and by the equation (17) we have

$$\begin{aligned} [0, \bar{Y} - \underline{X}] &\leq [0, \bar{Y} - \underline{Y} + \bar{X} - \underline{X}] \\ m_{ei}(X, Y) &\leq [0, \text{Diam}(Y) + \text{Diam}(X)]. \end{aligned} \quad (18)$$

By the equation (18) we conclude that if $m_{ei}(X, Y)$ satisfies the conditions of proposition 7 then it satisfies also the conditions of proposition 13.

Suppose that $m_{ei}(X, Y)$ satisfies the conditions of the proposition 8 then we get

$$m_{ei}(X, Y) = [0, \max\{(\underline{Y} - \bar{X}), (\bar{Y} - \underline{X})\}]. \quad (19)$$

By the hypothesis that $m_{ei}(X, Y)$ satisfies the conditions of proposition 8 and by the equation (19) we have

$$\begin{aligned} [0, \max\{(\underline{Y} - \bar{X}), (\bar{Y} - \underline{X})\}] &\leq [0, \bar{Y} - \underline{Y} + \bar{X} - \underline{X}] \\ m_{ei}(X, Y) &\leq [0, \text{Diam}(Y) + \text{Diam}(X)]. \end{aligned} \quad (20)$$

By the equation (20) we conclude that if $m_{ei}(X, Y)$ satisfies the conditions of proposition 8, then it also satisfies the conditions of proposition 13.

By the equations (16), (18) and (20) we conclude that $m_{ei}(X, Y) \leq [0, \text{Diam}(Y) + \text{Diam}(X)]$ if only if $X \cap Y \neq \emptyset$. ■

Proposition 14. Let X and $Y \in \mathbb{IR}$. Then $[0, \text{Diam}(Y) + \text{Diam}(X)] \leq m_{ei}(X, Y)$ if only if $X \cap Y = \emptyset$.

Proof. By the equations (16), (18) and (20) we conclude only in the case that $m_{ei}(X, Y)$ satisfies the conditions of propositions 6 and 14. Then $[0, \text{Diam}(Y) + \text{Diam}(X)] \leq m_{ei}(X, Y)$ if only if $X \cap Y = \emptyset$. ■

If we associate the uncertainty degree of an interval to its diameter, we can observe that the metric m_{ei} preserves the uncertainties, since a distance between two accurate intervals (null diameter) is an accurate measurement also of null diameter and the distance between two inaccurate intervals (diameter $\neq 0$) is also an inaccurate measurement, as it is shown in the following proposition.

Proposition 15. Let X and $Y \in \mathbb{IR}$, then we have $\text{Diam}(m_{ei}(X, Y)) \leq \text{Diam}(X) + \text{Diam}(Y)$.

Proof. We split this proof in two parts. The first one, in the case that $m_{ei}(X, Y)$ satisfies the conditions of proposition 6 and the second, the cases in which it satisfies the conditions of proposition 7 or the proposition 8, which can be directly inferred from propositions 12 and 13. For the first part, suppose that $m_{ei}(X, Y)$ satisfies the conditions of proposition 6, then we get

$$m_{ei}(X, Y) = [\underline{Y} - \overline{X}, \overline{Y} - \underline{X}]. \quad (21)$$

Therefore,

$$\begin{aligned} \text{Diam}(m_{ei}(X, Y)) &= (\overline{Y} - \underline{X}) - (\underline{Y} - \overline{X}) \\ &= \overline{Y} - \underline{X} - \underline{Y} + \overline{X} \\ &= (\overline{X} - \underline{X}) + (\overline{Y} - \underline{Y}) \\ &= \text{Diam}(X) + \text{Diam}(Y). \end{aligned} \quad (22)$$

With the proposed metric the module notion can be redefined as following.

Definition 17. Let X be an interval. The **interval module**, denoted by $|X|_I$, is defined by the distance $m_{ei}(X, [0; 0])$.

Theorem 5 (Interval Module Properties).

1. $|X|_I = 0 \Leftrightarrow X = 0$;
2. $|X + Y|_I \leq |X|_I + |Y|_I$;
3. $|X \cdot Y|_I = |X|_I \cdot |Y|_I$.

Proof. Property 1

Suppose that $X \neq 0$, then $\sup\{d_e(x,0) : x \in X\}$ is greater than zero. Therefore, $|X|_I \neq 0$.

Property 2

For that we need to prove that the inferior limit of $|X + Y|_I$ is always lesser than or equal to the inferior limit of $|X|_I + |Y|_I$ and that the superior limit of $|X + Y|_I$ is always lesser than or equal to the superior limit of $|X|_I + |Y|_I$. Thus we have

$$|X + Y|_I = |[X + Y; \overline{X} + \overline{Y}]|_I. \quad (23)$$

If $0 \in X + Y$, then

$$|X + Y|_I = [0; \max\{|X + Y|, |\overline{X} + \overline{Y}|\}]. \quad (24)$$

Case $0 \notin X + Y$, then

$$|X + Y|_I = [\min\{|X + Y|, |\overline{X} + \overline{Y}|\}; \max\{|X + Y|, |\overline{X} + \overline{Y}|\}], \quad (25)$$

and for $|X|_I + |Y|_I$ we get

$$|X|_I + |Y|_I = [|X|_I; \overline{|X|_I}] + [|Y|_I; \overline{|Y|_I}]. \quad (26)$$

If $0 \in X$ or Y , and not both. By assuming that $0 \in Y$, the other case is symmetrical, and thus, we have

$$|X|_I + |Y|_I = [\min\{|X|, |\overline{X}|\}; \max\{|X| + |\underline{Y}|, |\underline{Y}| + |\overline{X}|, |\underline{X}| + |\overline{Y}|, |\overline{X}| + |\overline{Y}|\}]. \quad (27)$$

If $0 \in X$ and Y , then

$$|X|_I + |Y|_I = [0; \max\{|X| + |\underline{Y}|, |\underline{Y}| + |\overline{X}|, |\underline{X}| + |\overline{Y}|, |\overline{X}| + |\overline{Y}|\}]. \quad (28)$$

Case $0 \notin X$ and Y , then

$$\begin{aligned} |X|_I + |Y|_I = & \\ & [\min\{|X| + |\underline{Y}|, |\underline{Y}| + |\overline{X}|, |\underline{X}| + |\overline{Y}|, |\overline{X}| + |\overline{Y}|\}; \\ & \max\{|X| + |\underline{Y}|, |\underline{Y}| + |\overline{X}|, |\underline{X}| + |\overline{Y}|, |\overline{X}| + |\overline{Y}|\}]. \end{aligned} \quad (29)$$

For the cases where $0 \in X + Y$ by the positive definition of the module it follows the inferior limit. It remains to prove the superior limit. For that it is enough to prove that

$$\max\{|X + Y|, |\overline{X} + \overline{Y}|\} \leq \max\{|X| + |\underline{Y}|, |\underline{Y}| + |\overline{X}|, |\underline{X}| + |\overline{Y}|, |\overline{X}| + |\overline{Y}|\}. \quad (30)$$

For that suppose that $\max\{|X + Y|, |\overline{X} + \overline{Y}|\} = |X + Y|$ then we have

$$|X + Y| \leq |X| + |\underline{Y}| \text{ as the real analysis.} \quad (31)$$

Now, assuming that $\max\{|X + Y|, |\overline{X} + \overline{Y}|\} = |\overline{X} + \overline{Y}|$ we get

$$|\overline{X} + \overline{Y}| \leq |\overline{X}| + |\overline{Y}|, \text{ as real analysis.} \quad (32)$$

So, when $0 \in X + Y$ we have $|X + Y|_I \leq |X|_I + |Y|_I$.

Now we analyse the case where $0 \notin X + Y$. In particular, for the case that X or Y contains zero and not both. As we see before, we chose Y containing the zeros and the other cases are symmetrical. We prove first for the inferior limit. For that we have to prove that

$$\min\{|X + Y|, |\overline{X} + \overline{Y}|\} \leq \min\{|X|, |\overline{X}|\}. \quad (33)$$

Suppose that $\min\{|\underline{X} + \underline{Y}|, |\overline{X} + \overline{Y}|\} = |\underline{X} + \underline{Y}|$, then $0 \leq X + Y$ and $\min\{|\underline{X}|, |\overline{X}|\} = |\underline{X}|$. As $\underline{Y} \leq 0$, we have $|\underline{X} + \underline{Y}| \leq |\underline{X}|$.

Now suppose that $\min\{|\underline{X} + \underline{Y}|, |\overline{X} + \overline{Y}|\} = |\overline{X} + \overline{Y}|$. Then $X + Y \leq 0$ and $\min\{|\underline{X}|, |\overline{X}|\} = |\overline{X}|$. As $0 \leq \overline{Y}$, it follows $|\overline{X} + \overline{Y}| \leq |\overline{X}|$, which proves the case for the inferior limit. The superior we can't get from the equations (30), (31) and (32).

In the case where $0 \notin X$ and Y , we will prove only for the inferior limit, since the superior limit follows from the equations (30), (31) and (32).

Suppose that $0 \leq X + Y$, then we have

$$\min\{|\underline{X} + \underline{Y}|, |\overline{X} + \overline{Y}|\} = |\underline{X} + \underline{Y}| \quad (34)$$

and

$$\min\{|\underline{X}| + |\underline{Y}|, |\underline{Y}| + |\overline{X}|, |\underline{X}| + |\overline{Y}|, |\overline{X}| + |\overline{Y}|\} = \begin{cases} |\underline{X}| + |\underline{Y}| & \text{if } 0 \leq X \text{ and } Y. \\ |\underline{Y}| + |\overline{X}| & \text{if } X \leq 0 \text{ and } 0 \leq Y. \\ |\underline{X}| + |\overline{Y}| & \text{if } Y \leq 0 \text{ and } 0 \leq X. \end{cases} \quad (35)$$

By the hypothesis that $0 \leq X + Y$ and by the conditions of the equation (35) we get

$$|\underline{X} + \underline{Y}| \leq \begin{cases} |\underline{X}| + |\underline{Y}| & \text{if } 0 \leq X \text{ and } Y. \\ |\underline{Y}| + |\overline{X}| & \text{if } X \leq 0 \text{ and } 0 \leq Y. \\ |\underline{X}| + |\overline{Y}| & \text{if } Y \leq 0 \text{ and } 0 \leq X. \end{cases} \quad (36)$$

So, we conclude the prove of the property 2, leaving the case where $X + Y \leq 0$ because it is similar to the cases where $0 \leq X + Y$.

Property 3

If X or Y contains zero we have

$$|X \cdot Y|_I = [0; \max\{|\underline{X} \cdot \underline{Y}|, |\underline{X} \cdot \overline{Y}|, |\overline{X} \cdot \underline{Y}|, |\overline{X} \cdot \overline{Y}|\}] \quad (37)$$

and

$$|X|_I \cdot |Y|_I = [0; \max\{|\underline{X}| \cdot |\underline{Y}|, |\underline{X}| \cdot |\overline{Y}|, |\overline{X}| \cdot |\underline{Y}|, |\overline{X}| \cdot |\overline{Y}|\}]. \quad (38)$$

As in the real analysis where the product of the module of two real numbers is equal to the module of the product of these numbers, in the interval module it is similar as it shown by the equations (37) and (38). So, given two intervals where at least one of them contains the zero, the interval module of the product is equal to the product of the modules of these numbers. Now, we will prove the property in discussion for the case where neither of the intervals contains zero.

If X and Y does not contain zero, we have

$$|X \cdot Y|_I = [\min\{|\underline{X} \cdot \underline{Y}|, |\underline{X} \cdot \overline{Y}|, |\overline{X} \cdot \underline{Y}|, |\overline{X} \cdot \overline{Y}|\}; \max\{|\underline{X} \cdot \underline{Y}|, |\underline{X} \cdot \overline{Y}|, |\overline{X} \cdot \underline{Y}|, |\overline{X} \cdot \overline{Y}|\}] \quad (39)$$

and

$$|X|_I \cdot |Y|_I = [\min\{|\underline{X}| \cdot |\underline{Y}|, |\underline{X}| \cdot |\overline{Y}|, |\overline{X}| \cdot |\underline{Y}|, |\overline{X}| \cdot |\overline{Y}|\}; \max\{|\underline{X}| \cdot |\underline{Y}|, |\underline{X}| \cdot |\overline{Y}|, |\overline{X}| \cdot |\underline{Y}|, |\overline{X}| \cdot |\overline{Y}|\}]. \quad (40)$$

The property 3 follows directly from the equivalence between the equations (39) and (40). ■

The figure 2 gives a geometric interpretation for the module of a Moore-interval.

Fig. 2. Geometric interpretation of the interval module of an interval

Theorem 6. Let $X, Y, Z \in \mathbb{IR}$ be intervals. Then, the following properties are true:

1. $m_{ei}(X + Y, X + Z) = m_{ei}(Y, Z) + m_{ei}(X, X)$ or $m_{ei}(Y, Z) + [-Diam(X); Diam(X)]$
2. $X \subseteq Y \Rightarrow |X|_I \subseteq |Y|_I$

Proof. **Item 1** If $m_{ei}(X + Y, X + Z)$ satisfies the conditions of proposition 6 we have

$$\begin{aligned}
 m_{ei}(X + Y, X + Z) &= [(\underline{X} + \underline{Z}) - (\overline{X} + \overline{Y}); (\overline{X} + \overline{Z}) - (\underline{X} + \underline{Y})] \\
 &= [(\underline{Z} - \overline{Y}) + (\underline{X} - \overline{X}); (\overline{Z} - \underline{Y}) + (\overline{X} - \underline{X})] \\
 &= [(\underline{Z} - \overline{Y}) - Diam(X); (\overline{Z} - \underline{Y}) + Diam(X)] \\
 &= [(\underline{Z} - \overline{Y}); (\overline{Z} - \underline{Y})] + [-Diam(X); Diam(X)] \\
 &= m_{ei}(Y, Z) + [-Diam(X); Diam(X)].
 \end{aligned} \tag{41}$$

If $m_{ei}(X + Y, X + Z)$ satisfies the conditions of corollary 3 we get

$$\begin{aligned}
 m_{ei}(X + Y, X + Z) &= [0; \max\{(\overline{X} + \overline{Y}) - (\underline{X} + \underline{Z}), (\overline{X} + \overline{Z}) - (\underline{X} + \underline{Y})\}] \\
 &= [0; \max\{(\overline{Y} - \underline{Z}) + (\overline{X} - \underline{X}), (\overline{Z} - \underline{Y}) + (\overline{X} - \underline{X})\}] \\
 &= [0; \max\{(\overline{Y} - \underline{Z}) + Diam(X), (\overline{Z} - \underline{Y}) + Diam(X)\}] \\
 &= [0; \max\{(\overline{Y} - \underline{Z}), (\overline{Z} - \underline{Y})\}] + [0; Diam(X)] \\
 &= m_{ei}(Y, Z) + [0; Diam(X)] \\
 &= m_{ei}(Y, Z) + m_{ei}(X, X).
 \end{aligned} \tag{42}$$

Item 2 It follows directly from propositions 16 and 17. ■

The notion of circumference, limit and continuity can be constructed by this metric which will appear in another work.

Finally, about the Moore metric can be said that in the implementation of the absolute value function for intervals of real numbers X , by Moore Moore (1979), in programming language, for example

$$|X| = \max[|\underline{X}|, |\overline{X}|]$$

gives a single scalar value, and not an interval.

When passing intervals to an algorithm that was originally written with other numerical types in mind some users have expressed surprise at the result returned by the $|X|$ function. They would expect, for instance, that the absolute value of the interval $[-3, 2]$ would return $[0, 3]$, which is a reasonable expectation, and would allow many more algorithms to work without modification. In some interval mathematics applications, such as non-smooth optimisation, you want " $| \cdot |$ " to be the range of the absolute value, whereas, in other implementations, you want it to be the "magnitude," that is, the largest absolute value, rounded up, of any point in the argument interval. Additionally, the "magnitude," or the smallest absolute value of

any point in the interval, rounded down, is usually included, for a triad of extensions to the absolute value. The magnitude is useful, for example, in proving diagonal dominance of a matrix with uncertain entries.

5. Conclusion

By it was said above the distance notion is ubiquitous to mainly research areas and for system modelling with uncertainties. The usual metrics fail to accomplish those necessities, because it is not indeed an interval metric. By other hand the proposed metric is not only a generalisation of the usual Euclidean metric but also it is an essentially interval metric, as it is was shown above. With the proposed metric one can redefine the notions of limit, continuity, neighbourhood, convergence etc. Most basic properties of mathematics can be redefined with the proposed notion of distance.

The proposed metric accomplishes the numeric aspect if one takes an interval as an approximation of a real number and the logic aspects if one takes an interval as a fuzzy information.

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