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Optimal control systems of second order with infinite time horizon - maximum principle[*]

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Abstract

In the paper, we consider an optimal control problem for a second order control system on unbounded interval $(0, \infty)$ and an integral cost functional.

In the first part of the paper, we recall some results concerning the existence and uniqueness of a solution to the control system, corresponding to any admissible control, the continuous dependence of solutions on controls and the existence of the so-called classically optimal solution to the optimal control problem under consideration. These results has been obtained in [D. Idczak, S. Walczak, *Optimal control systems of second order with infinite time horizon - existence of solutions*, to appear in JOTA].

In the second part, some other definitions of optimality are introduced and their interrelationships, including optimality principle, are given. Two maximum principles stating necessary conditions for the introduced kinds of optimality (in general case and in a special one) are derived.

1. Introduction

As in Idczak (to appear), we consider a control system described by the following system of the second order equations

$$\ddot{x}(t) = G_x(t, x(t), u(t)), \ t \in I := (0, \infty) \ a.e, \tag{1}$$

with the initial condition

$$x(0) = 0, \tag{2}$$

where $G_x : I \times \mathbb{R}^n \times M \to \mathbb{R}^n$ is the gradient with respect to x of a function $G : I \times \mathbb{R}^n \times M \to \mathbb{R}$, $M \subset \mathbb{R}^m$ is a fixed set. In the next, we shall use notations and definitions introduced in Idczak (to appear). In particular, we assume that the controls $u(\cdot)$ belong to a set $\mathcal{U}^p := \{u \in L^p(I, \mathbb{R}^m); u(t) \in M \text{ for } t \in I \text{ a.e.}\}$, $p \in [1, \infty]$, and the trajectories $x(\cdot)$ - to Sobolev space $H_0^1(I, \mathbb{R}^n)$ (cf. Brezis (1983)). Since each function $x(\cdot) \in H^1(I, \mathbb{R}^n)$ possesses the limit $\lim_{t \to \infty} x(t) = 0$, therefore the problem of the existence of a solution to (1)-(2) in the space

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$H^1_0(I, \mathbb{R}^n)$, corresponding to a control $u(\cdot)$, is, in fact, two-point boundary value problem with boundary conditions

$$x(0) = 0, \; x(\infty) := \lim_{t \to \infty} x(t) = 0.$$

We say that a function $x_u(\cdot) \in H^1_0(I, \mathbb{R}^n)$ is a solution to (1)-(2), corresponding to control $u(\cdot)$, if

$$\int_I (\langle \dot{x}_u(t), \dot{h}(t) \rangle + \langle G_x(t, x_u(t), u(t)), h(t) \rangle) dt = 0 \tag{3}$$

for any $h(\cdot) \in H^1_0(I, \mathbb{R}^n)$.

The study of systems (1) in the space $H^1(I, \mathbb{R}^n)$ is justified because in this space both kinetic energy $E_k = \frac{1}{2} \int_I \left| \dot{x}_u(t) \right|^2 dt$ and potential one $\int_I G(t, x_u(t), u(t)) dt$ of the system are finite as in the real world (in the case of potential energy - under appropriate assumptions on G). Potential form of the right-hand side of this system allows us to use a variational approach.

In the case of bounded time interval I (finite horizon), classical cost functional for optimal control problems has the following integral form

$$J(x, u) = \int_I f(t, x(t), u(t)) dt.$$

When $I = (0, \infty)$ (infinite horizon) assumptions guarantying the integrability of the function $I \ni t \longmapsto f(t, x(t), u(t)) \in \mathbb{R}$ are often too restrictive and they are not fulfilled in some (e.g. economical) applications. So, in such a case it is necessary to consider some other concepts of optimality. Following Carlson and Haurie (cf. Carlson (1987)) we use the notions of strong, catching-up, sporadically catching-up and finitely optimal solution to the optimal control problem under consideration and show their interrelationships. A review of the concepts of optimality for the first order problems with infinite horizon and their interrelationships are given in Carlson (1989).

In the first part of the paper, we recall main results concerning system (1)-(2), obtained in Id-czak (to appear), namely, on the existence and uniqueness of a solution $x_u(\cdot) \in H^1_0(I, \mathbb{R}^n)$ to (1)-(2), corresponding to any control $u(\cdot) \in U^p$, and its continuous dependence on $u(\cdot)$ (Theorems 1, 2, 4). Next, we recall the existence results for an optimal control problem connected with (1)-(2) and a cost functional of integral type, obtained in Idczak (to appear) (Theorems 5, 6).

In the second - main part of the paper, we derive necessary conditions for optimality in the sense of the mentioned notions of optimality. Theorem 13 concerns a general form of cost functional. The proof of this theorem is based on the so called optimality principle (Theorem 9) and the maximum principle for finite horizon second order optimal control problems, obtained in Idczak (1998). Theorem 15 concerns some special case of cost functional. Proof of this theorem is based on Theorem 13.

The appropriate optimality principle and necessary optimality conditions for the first order systems with infinite horizon have been obtained in Halkin (1974) (cf. also (Carlson, 1987, Th. 2.3)).

2. Existence, uniqueness and stability

Let us denote $B_{\mathbb{R}^n}(0, r) = \{x \in \mathbb{R}^n; \; |x| \leq r\}$, $B_{H^1_0(I, \mathbb{R}^n)}(0, r) = \{x(\cdot) \in H^1_0(I, \mathbb{R}^n); \; \|x(\cdot)\|_{H^1_0(I, \mathbb{R}^n)} \leq r\}$ and formulate the following assumptions:

A1a. *function $G(t,\cdot,\cdot) : \mathbb{R}^n \times M \to \mathbb{R}$ is continuous for $t \in I$ a.e. and function $G(\cdot,x,u) : I \to \mathbb{R}$ is measurable in Lebesgue sense for any $(x,u) \in \mathbb{R}^n \times M$*

A1b. *function $G_x(t,\cdot,\cdot) : \mathbb{R}^n \times M \to \mathbb{R}^n$ is continuous for $t \in I$ a.e. and function $G_x(\cdot,x,u) : I \to \mathbb{R}^n$ is measurable in Lebesgue sense for any $(x,u) \in \mathbb{R}^n \times M$*

A2. *there exist constants $b_2 > 0$, $c_2 > 0$, functions $b_1(\cdot), c_1(\cdot) \in L^2(I,\mathbb{R})$ and $b_0(\cdot), c_0(\cdot) \in L^1(I,\mathbb{R})$ such that*

$$b_2 |x|^2 + b_1(t) |x| + b_0(t) \le G(t,x,u) \le c_2 |x|^2 + c_1(t) |x| + c_0(t)$$

for $t \in I$ a.e., $x \in \mathbb{R}^n$, $u \in M$

A3. *for any $r > 0$ there exist a constant $d_1 > 0$ and a function $d_0(\cdot) \in L^2(I,\mathbb{R})$ such that*

$$|G_x(t,x,u)| \le d_1 |x| + d_0(t)$$

for $t \in I$ a.e., $x \in B_{\mathbb{R}^n}(0,r)$, $u \in M$.

By r_0 we mean a constant

$$r_0 = \frac{\overline{b}_1 + \sqrt{\overline{b}_1^2 - 4\overline{b}_2(\overline{b}_0 - \overline{c}_0)}}{2\overline{b}_2}$$

where $\overline{b}_2 = \min\{\frac{1}{2}, b_2\}$, $\overline{b}_1 = (\int_I |b_1(t)|^2 \, dt)^{\frac{1}{2}}$, $\overline{b}_0 = \int_I b_0(t) dt$, $\overline{c}_0 = \int_I c_0(t) dt$. This constant is always nonnegative. Since the case of $r_0 = 0$ is not interesting (in such a case the zero function is the unique (in $H_0^1(I,\mathbb{R}^n)$) solution to (1)-(2) for any control $u(\cdot) \in \mathcal{U}^p$ (cf. (Idczak, to appear, proof of Theorem 5))), therefore we shall assume in the next that $r_0 > 0$.

In the next, we shall also use the following two assumptions

A4a. *function $G(t,\cdot,u) : B_{\mathbb{R}^n}(0,r_0) \to \mathbb{R}$ is convex for $t \in I$ a.e. and $u \in M$,*

A4b. *function $G(t,\cdot,u) : \mathbb{R}^n \to \mathbb{R}$ is convex for $t \in I$ a.e. and $u \in M$*

We have

Theorem 1. *If G satisfies assumptions A1a, A1b, A2, A3 and A4a, then, for any fixed $u(\cdot) \in \mathcal{U}^p$, there exists a solution $x_u(\cdot) \in B_{H_0^1(I,\mathbb{R}^n)}(0,r_0)$ to (1)-(2) which is unique in $B_{H_0^1(I,\mathbb{R}^n)}(0,r_0)$. If, additionally, G satisfies A4b, then the solution $x_u(\cdot)$ is unique in $H_0^1(I,\mathbb{R}^n)$.*

2.1 Stability - case of strong convergence of controls

Let us assume that G is lipschitzian with respect to $u \in M$, i.e.

A5. *there exists a function $k(\cdot) \in L^q(I,\mathbb{R}^n)$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that*

$$|G(t,x,u_1) - G(t,x,u_2)| \le k(t) |u_1 - u_2|$$

for $t \in I$ a.e., $x \in B_{\mathbb{R}^n}(0,r_0)$, $u_1, u_2 \in M$.

We have

Theorem 2. *Let G satisfies assumptions A1a, A1b, A2, A3, A4a and A5. If a sequence of controls $(u_k(\cdot))_{k\in\mathbb{N}} \subset \mathcal{U}^p$ converges in $L^p(I,\mathbb{R}^m)$ to $u_0(\cdot) \in \mathcal{U}^p$ with respect to the norm topology, then the sequence $(x_k(\cdot))_{k\in\mathbb{N}}$ of corresponding solutions to (1)-(2), belonging to $B_{H_0^1(I,\mathbb{R}^n)}(0,r_0)$, converges weakly in $H_0^1(I,\mathbb{R}^n)$ to a solution $x_0(\cdot)$ to (1)-(2), corresponding to the control $u_0(\cdot)$ and belonging to $B_{H_0^1(I,\mathbb{R}^n)}(0,r_0)$.*

Remark 3. *Weak convergence of a sequence $(x_k(\cdot))_{k \in \mathbb{N}}$ in $H_0^1(I, \mathbb{R}^n)$ implies (cf. Lieb (1997)) its uniform convergence on any finite interval $[0, T] \subset I$ and weak convergence of sequences $(x_k(\cdot))_{k \in \mathbb{N}}$, $(\dot{x}_k(\cdot))_{k \in \mathbb{N}}$ in $L^2(I, \mathbb{R}^n)$.*

2.2 Stability - case of $p = \infty$ and weak-* convergence of controls

Now, we shall consider the set of controls

$$\mathcal{U}^\infty = \{u \in L^\infty(I, \mathbb{R}^m); \ u(t) \in M \text{ for } t \in I \text{ a.e.}\}$$

with the weak-* topology induced from $L^\infty(I, \mathbb{R}^m)$. We assume that function G is affine in u, i.e.

$$G(t, x, u) = G^1(t, x) + G^2(t, x)u \tag{4}$$

where functions $G^1 : I \times \mathbb{R}^n \to \mathbb{R}$, $G^2 : I \times \mathbb{R}^n \to \mathbb{R}^m$ are measurable in $t \in I$, continuous in $x \in \mathbb{R}^n$ and

A6. *there exists a function $\gamma \in L^1(I, \mathbb{R})$ such that*

$$\left| G^2(t, x) \right| \leq \gamma(t)$$

for $t \in I$ a.e., $x \in B_{\mathbb{R}^n}(0, r_0)$.

We have

Theorem 4. *Let G of the form (4) satisfies assumptions A1a, A1b, A2, A3, A4a and A6. If a sequence of controls $(u_k(\cdot))_{k \in \mathbb{N}} \subset \mathcal{U}^\infty$ converges in $L^\infty(I, \mathbb{R}^m)$ to $u_0(\cdot) \in \mathcal{U}^\infty$ with respect to the weak-* topology, then the sequence $(x_k(\cdot))_{k \in \mathbb{N}}$ of corresponding solutions to (1)-(2), belonging to $B_{H_0^1(I, \mathbb{R}^n)}(0, r_0)$, converges in the weak topology of $H_0^1(I, \mathbb{R}^n)$ to the solution $x_0(\cdot)$ to (1)-(2), corresponding to $u_0(\cdot)$ and belonging to $B_{H_0^1(I, \mathbb{R}^n)}(0, r_0)$.*

3. Optimal control - existence of solutions

3.1 Affine case

Let us consider control system

$$\ddot{x}(t) = G_x^1(t, x(t)) + G_x^2(t, x(t))u(t), \ t \in I \text{ a.e,} \tag{5}$$

with cost functional

$$J(x(\cdot), u(\cdot)) = \int_I (\langle \alpha(t), \dot{x}(t) \rangle + f(t, x(t), u(t)))dt \to \min. \tag{6}$$

By *a classical solution* to problem (5)-(6) in the set $B_{H_0^1(I, \mathbb{R}^n)}(0, r_0) \times \mathcal{U}^\infty$ $(H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty)$ we mean a pair $(x_0(\cdot), u_0(\cdot)) \in B_{H_0^1(I, \mathbb{R}^n)}(0, r_0) \times \mathcal{U}^\infty$ $((x_0(\cdot), u_0(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty)$ satisfying system (5) and such that

$$J(x_0(\cdot), u_0(\cdot)) \leq J(x(\cdot), u(\cdot)) \tag{7}$$

for any pair $(x(\cdot), u(\cdot)) \in B_{H_0^1(I, \mathbb{R}^n)}(0, r_0) \times \mathcal{U}^\infty$ $((x(\cdot), u(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty)$ satisfying system (5).

We assume that

A7. *a set $M \subset \mathbb{R}^m$ and functions $\alpha : I \to \mathbb{R}^n$, $f : I \times B_{\mathbb{R}^n}(0, r_0) \times M \to \mathbb{R}$ are such that*

a) *M is convex and compact*

b) $\alpha(\cdot) \in L^2(I, \mathbb{R}^n)$

c) *function f is $\mathcal{L}(I) \otimes \mathcal{B}(B_{\mathbb{R}^n}(0, r_0)) \otimes \mathcal{B}(M)$ - measurable ($\mathcal{L}(I)$ means the σ-field of Lebesgue measurable subsets of I, $\mathcal{B}(B_{\mathbb{R}^n}(0, r_0))$, $\mathcal{B}(M)$ - the σ-fields of Borel subsets of $B_{\mathbb{R}^n}(0, r_0)$, M, respectively; $\mathcal{L}(I) \otimes \mathcal{B}(B_{\mathbb{R}^n}(0, r_0)) \otimes \mathcal{B}(M)$ is the product σ-field)*

d) *for $t \in I$ a.e. function $f(t, \cdot, \cdot)$ is lower semicontinuous on $B_{\mathbb{R}^n}(0, r_0) \times M$*

e) *for $t \in I$ a.e. and any $x \in B_{\mathbb{R}^n}(0, r_0)$, function $f(t, x, \cdot)$ is convex on M*

f) *there exists a function $\beta \in L^1(I, \mathbb{R})$ such that*

$$|f(t, x, u)| \le \beta(t)$$

for $t \in I$ a.e., $x \in B_{\mathbb{R}^n}(0, r_0)$, $u \in M$.

If the function G of the form (4) satisfies assumptions of Theorem 1 and A7 is fulfilled, then for any control $u(\cdot) \in \mathcal{U}^\infty$ there exists a unique in $B_{H_0^1(I, \mathbb{R}^n)}(0, r_0)$ solution $x_u(\cdot)$ of control system (5) and cost functional (6) has a finite value at the pair $(x_u(\cdot), u(\cdot))$ (if G satisfies A4b, then the solution $x_u(\cdot)$ is unique in $H_0^1(I, \mathbb{R}^n)$).
Moreover (cf. Idczak (to appear)), we have

Theorem 5. *If G of the form (4) satisfies assumptions A1a, A1b, A2, A3, A4a, A6 and assumption A7 is satisfied, then optimal control problem (5)-(6) has a classical solution in the set $B_{H_0^1(I, \mathbb{R}^n)}(0, r_0) \times \mathcal{U}^\infty$. If, additionally, assumption A4b is satisfied, then problem (5)-(6) has a classical solution in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$.*

In Idczak (to appear), an example illustrating the above theorem is given.

3.2 Nonlinear case

Now, let us consider the nonlinear system (1) with cost functional (6). Below, by $\mathcal{U}_0^p$ ($p \in [1, \infty]$) we mean a compact (in norm topology of $L^p(I, \mathbb{R}^m)$) set of controls, contained in $\mathcal{U}^p$. We have

Theorem 6. *If G satisfies assumptions A1a, A1b, A2, A3, A4a, A5 and assumption A7 is satisfied, then optimal control problem (1)-(6) has a classical solution in the set $B_{H_0^1(I, \mathbb{R}^n)}(0, r_0) \times \mathcal{U}_0^p$. If, additionally, assumption A4b is satisfied, then problem (1)-(6) has a classical solution in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}_0^p$.*

Remark 7. *Definition of a classical solution to problem (1)-(6) in the set $B_{H_0^1(I, \mathbb{R}^n)}(0, r_0) \times \mathcal{U}_0^p$ is quite analogous as in the case of the problem (5)-(6) and the set $B_{H_0^1(I, \mathbb{R}^n)}(0, r_0) \times \mathcal{U}^\infty$ (with (5) replaced by (1) and $\mathcal{U}^\infty$ replaced by $\mathcal{U}_0^p$ - cf. section 3.2).*

4. Optimal control - optimality principle

In this section we assume that assumptions A1a, A1b, A2, A3 and A7 are satisfied.
Since in the next we shall consider system (1) also on a finite time interval, therefore below we give a definition of a solution to such a finite horizon system (cf. Idczak (1998)).
We say that a pair $(x_u(\cdot), u(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^\infty$ satisfies system (1) a.e. on $(0, T)$ if

$$\int_0^T \left\langle \dot{x}_u(t), \dot{h}(t) \right\rangle + \langle G_x(t, x_u(t), u(t)), h(t) \rangle \, dt = 0 \tag{8}$$

for any $h(\cdot) \in H_0^1((0,T), \mathbb{R}^n)$, where $H_0^1((0,T), \mathbb{R}^n)$ is the classical Sobolev space of functions $h(\cdot) \in H^1((0,T), \mathbb{R}^n)$ satisfying the boundary conditions

$$h(0) = h(T) = 0$$

and $\mathcal{U}_{(0,T)}^\infty := \{u \in L^\infty((0,T), \mathbb{R}^m); \; u(t) \in M \text{ for } t \in I \text{ a.e.}\}$.
By J_T we shall mean the functional given by the formula

$$J_T(x(\cdot), u(\cdot)) = \int_0^T (\langle \alpha(t), \dot{x}(t) \rangle + f(t, x(t), u(t))) dt.$$

In the theory of infinite horizon optimal control the following concepts of optimality, different from the classical one (cf. (7)), are used (cf. (Carlson, 1987, Definition 1.2), Carlson (1989)).
By *a strong solution* of the problem (1)-(6) in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ we mean a pair $(x_0(\cdot), u_0(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ which satisfies system (1) and

$$\lim_{T \to \infty} (J_T(x_0(\cdot), u_0(\cdot)) - J_T(x(\cdot), u(\cdot))) \leq 0$$

for any pair $(x(\cdot), u(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ satisfying system (1).
By *a catching-up solution* of the problem (1)-(6) in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ we mean a pair $(x_0(\cdot), u_0(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ which satisfies system (1) and

$$\limsup_{T \to \infty} (J_T(x_0(\cdot), u_0(\cdot)) - J_T(x(\cdot), u(\cdot))) \leq 0$$

for any pair $(x(\cdot), u(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ satisfying system (1). This is equivalent to the following condition: for any pair $(x(\cdot), u(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ satisfying system (1) and any $\varepsilon > 0$ there exists $T_0 > 0$ such that

$$J_T(x_0(\cdot), u_0(\cdot)) - J_T(x(\cdot), u(\cdot)) < \varepsilon$$

for $T > T_0$.
By *a sporadically catching-up solution* of the problem (1)-(6) in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ we mean a pair $(x_0(\cdot), u_0(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ which satisfies system (1) and

$$\liminf_{T \to \infty} (J_T(x_0(\cdot), u_0(\cdot)) - J(x(\cdot), u(\cdot))) \leq 0$$

for any pair $(x(\cdot), u(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ satisfying system (1). This is equivalent to the following condition: for any pair $(x(\cdot), u(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ satisfying system (1), any $\varepsilon > 0$ and any $T > 0$ there exists $T' > T$ such that

$$J_{T'}(x_0(\cdot), u_0(\cdot)) - J_{T'}(x(\cdot), u(\cdot)) < \varepsilon.$$

By *a finitely optimal solution* of problem (1)-(6) we mean a pair $(x_0(\cdot), u_0(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ satisfying (1) a.e. on I and such that for any $T > 0$ and any pair $(x(\cdot), u(\cdot)) \in H^1((0,T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^\infty$ satisfying system (1) a.e. on $(0,T)$ and boundary conditions

$$x(0) = 0, \; x(T) = x_0(T),$$

we have

$$J_T(x_0(\cdot), u_0(\cdot)) \leq J_T(x(\cdot), u(\cdot)).$$

It is obvious (under our assumptions) that classical optimality implies the strong one, strong optimality implies the catching-up one, catching-up optimality implies the sporadically catching-up one. In the next theorem we shall show that sporadically catching-up optimality implies the finite one. Before we prove this theorem we shall prove the following technical lemma.

Lemma 8. *If a function $x(\cdot) \in H^1((0,T),\mathbb{R}^n)$ is such that $x(0) = 0$, $v(\cdot) \in H^1_0(I,\mathbb{R}^n)$ and $x(T) = v(T)$, then the function*

$$z : I \ni t \mapsto \left\{ \begin{array}{l} x(t) \text{ for } t \in (0,T] \\ v(t) \text{ for } t \in (T,\infty) \end{array} \right. \in \mathbb{R}^n$$

belongs to $H^1_0(I,\mathbb{R}^n)$ and its weak derivative g has the form

$$g : I \ni t \mapsto \left\{ \begin{array}{l} \dot{x}(t) \text{ for } t \in (0,T] \\ \dot{v}(t) \text{ for } t \in (T,\infty) \end{array} \right. \in \mathbb{R}^n.$$

Proof. First of all, let us point that $z(\cdot), g(\cdot) \in L^2(I,\mathbb{R}^n)$. Next, let us define the function

$$y : I \ni t \mapsto \int_T^t g(\tau)d\tau + x(T) \in \mathbb{R}^n.$$

Of course,

$$y(0) = \int_T^0 g(\tau)d\tau + x(T) = -\int_0^T g(\tau)d\tau + x(T) = -\int_0^T \dot{x}(\tau)d\tau + x(T) = -x(T) + x(T) = 0.$$

Moreover, $y(\cdot) \in L^1_{loc}(I,\mathbb{R}^n)$ and from (Brezis, 1983, Lemma VIII.2)

$$\int_0^\infty y(\tau)\dot{\varphi}(\tau)d\tau = \int_0^\infty (\int_T^\tau g(s)ds)\dot{\varphi}(\tau)d\tau + \int_0^\infty x(T)\dot{\varphi}(\tau)d\tau$$
$$= \int_0^\infty (\int_T^\tau g(s)ds)\dot{\varphi}(\tau)d\tau = -\int_0^\infty g(\tau)\varphi(\tau)d\tau$$

for any $\varphi(\cdot) \in C^1_c(I,\mathbb{R}^n)$ (the space of continuously differentiable functions $\varphi : I \to \mathbb{R}^n$ with compact support $\text{supp}\varphi \subset I$). This means that the weak derivative of $y(\cdot)$ exists and is equal to the function $g(\cdot)$. Now, we shall show that $y(\cdot) = z(\cdot)$. Indeed, if $t_0 \in (0,T)$, then

$$y(t_0) = \int_T^{t_0} g(\tau)d\tau + x(T) = -\int_{t_0}^T g(\tau)d\tau + x(T) = -(\int_0^T g(\tau)d\tau - \int_0^{t_0} g(\tau)d\tau) + x(T)$$
$$= -\int_0^T \dot{x}(\tau)d\tau + \int_0^{t_0} \dot{x}(\tau)d\tau + x(T) = -x(T) + x(t_0) + x(T) = x(t_0) = z(t_0).$$

If $t_0 \in (T,\infty)$, then

$$y(t_0) = \int_T^{t_0} g(\tau)d\tau + x(T) = \int_T^{t_0} \dot{v}(\tau)d\tau + v(T) = \int_T^{t_0} \dot{v}(\tau)d\tau + \int_0^T \dot{v}(\tau)d\tau = \int_0^{t_0} \dot{v}(\tau)d\tau$$
$$= v(t_0) = z(t_0).$$

Of course, $y(T) = x(T) = z(T)$. Thus, $z(\cdot) \in H^1_0(I,\mathbb{R}^n)$. ∎

Now, we are in the position to prove the following optimality principle that is analogous to the appropriate result for infinite horizon first order optimal control problems with initial conditions (cf. (Carlson, 1987, Theorem 2.2)).

Theorem 9 (optimality principle). *If a pair $(x_0(\cdot), u_0(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ is a sporadically catching-up solution to (1)-(6) in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$, then it is finitely optimal solution to this problem in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$.*

Proof. Let us suppose that for some $T > 0$ there exists a pair $(x_*(\cdot), u_*(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^\infty$ satisfying system (1) a.e. on $[0, T]$ and such that

$$x_*(0) = 0, \ x_*(T) = x_0(T)$$

and

$$J_T(x_*(\cdot), u_*(\cdot)) < J_T(x_0(\cdot), u_0(\cdot)). \tag{9}$$

Let us define a pair $(x_+(\cdot), u_+(\cdot))$ in the following way

$$x_+ : I \ni t \mapsto \begin{cases} x_*(t) \text{ for } t \in (0, T] \\ x_0(t) \text{ for } t \in (T, \infty) \end{cases} \in \mathbb{R}^n.$$

$$u_+ : I \ni t \mapsto \begin{cases} u_*(t) \text{ for } t \in (0, T] \\ u_0(t) \text{ for } t \in (T, \infty) \end{cases} \in \mathbb{R}^n.$$

Lemma 8 implies that $x_+(\cdot) \in H_0^1(I, \mathbb{R}^n)$. Of course, $u_+(\cdot) \in \mathcal{U}^\infty$. Now, we shall check that the pair $(x_+(\cdot), u_+(\cdot))$ satisfies system (1). Since the pair $(x_*(\cdot), u_*(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^\infty$ satisfies system (1), therefore the function $x_*(\cdot)$ possesses the classical second order derivative $\ddot{x}_*(t)$ for $t \in (0, T)$ a.e. and

$$\ddot{x}_*(t) = G_x(t, x_*(t), u_*(t)), \ t \in (0, T) \text{ a.e.}$$

In the same way, the function $x_0(\cdot)$ possesses the classical second order derivative $\ddot{x}_0(t)$ for $t \in I$ a.e. and

$$\ddot{x}_0(t) = G_x(t, x_0(t), u_0(t)), \ t \in I \text{ a.e.}$$

Consequently, the function $x_+(\cdot)$ possesses the classical second order derivative $\ddot{x}_+(t)$ for $t \in I$ a.e. and

$$\ddot{x}_+(t) = G_x(t, x_+(t), u_+(t)), \ t \in I \text{ a.e.}$$

A3 implies that $\ddot{x}_+(\cdot) \in L^2(I, \mathbb{R}^n)$. Thus, for any $h(\cdot) \in H_0^1(I, \mathbb{R}^n)$, we have

$$\int_I \langle \ddot{x}_+(t), h(t) \rangle dt = \int_I \langle G_x(t, x_+(t), u_+(t)), h(t) \rangle dt. \tag{10}$$

The function $\dot{x}_*(\cdot)$ (more precisely, its continuous representant) is absolutely continuous on $[0, T]$ (cf. (Brezis, 1983, Theorem VIII.2)). Also, the function $\dot{x}_0(\cdot)$ is absolutely continuous on each compact subinterval of $[T, \infty)$. Consequently, the function $\dot{x}_+(\cdot)$ is absolutely continuous on each compact subinterval of I. So, integrating by parts we obtain

$$\int_I \langle \ddot{x}_+(t), h(t) \rangle dt = \lim_{P \to \infty} \int_0^P \langle \ddot{x}_+(t), h(t) \rangle dt = \lim_{P \to \infty} (\langle \dot{x}_+(P), h(P) \rangle - \langle \dot{x}_+(0), h(0) \rangle$$

$$- \int_0^P \langle \dot{x}_+(t), \dot{h}(t) \rangle dt) = -\lim_{P \to \infty} \int_0^P \langle \dot{x}_+(t), \dot{h}(t) \rangle dt = -\int_I \langle \dot{x}_+(t), \dot{h}(t) \rangle dt$$

for any $h(\cdot) \in H_0^1(I, \mathbb{R}^n)$ (we used here the fact that $\lim_{P\to\infty} \dot{x}_+(P) = \lim_{P\to\infty} \dot{x}_0(P) = 0$, which follows from Lemma 8 and the relation $\dot{x}_0(\cdot) \in H^1(I, \mathbb{R}^n)$). Putting this value to (10) we obtain

$$\int_I (\left\langle \dot{x}_+(t), \dot{h}(t) \right\rangle + \langle G_x(t, x_+(t), u_+(t)), h(t) \rangle) dt = 0$$

for any $h(\cdot) \in H_0^1(I, \mathbb{R}^n)$. This means that the pair $(x_+(\cdot), u_+(\cdot))$ satisfies (1).
Now, from (9) it follows that there exists $\varepsilon > 0$ such that

$$J_T(x_*(\cdot), u_*(\cdot)) + \varepsilon < J_T(x_0(\cdot), u_0(\cdot)).$$

From the other hand, sporadically catching-up optimality of the pair $(x_0(\cdot), u_0(\cdot))$ implies that there exists $Q > T$ such that

$$J_Q(x_0(\cdot), u_0(\cdot)) < \frac{\varepsilon}{2} + J_Q(x_+(\cdot), u_+(\cdot)).$$

But

$$\frac{\varepsilon}{2} + J_Q(x_+(\cdot), u_+(\cdot)) = \frac{\varepsilon}{2} + J_T(x_*(\cdot), u_*(\cdot)) + \int_T^Q (\left\langle \alpha(t), \dot{x}_0(t) \right\rangle + f(t, x_0(t), u_0(t))) dt$$

$$< J_T(x_0(\cdot), u_0(\cdot)) - \frac{\varepsilon}{2} + \int_T^Q (\left\langle \alpha(t), \dot{x}_0(t) \right\rangle + f(t, x_0(t), u_0(t))) dt = J_Q(x_0(\cdot), u_0(\cdot)) - \frac{\varepsilon}{2}.$$

The obtained contradiction completes the proof. ∎

5. Optimal control - maximum principle

In Idczak (1998) a maximum principle for the following finite horizon optimal control problem has been obtained:

$$\frac{d}{dt}(H_{\dot{z}}(t, z(t), \dot{z}(t), u(t))) = H_z(t, z(t), \dot{z}(t), u(t)), \ t \in (0, T) \text{ a.e.,} \tag{11}$$

$$z(0) = z(T) = 0, \tag{12}$$

$$I_T(z(\cdot), u(\cdot)) = \int_0^T H_0(t, z(t), \dot{z}(t), u(t)) dt \to \min. \tag{13}$$

where $H, H_0 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, z(\cdot) \in H_0^1((0, T), \mathbb{R}^n), u(\cdot) \in \mathcal{U}_{(0,T)}^\infty, T > 0$ is fixed.

Remark 10. *In fact, in Idczak (1998) the time interval $(0, \pi)$ was considered. Of course, it may be replaced by $(0, T)$ with any $T > 0$. In such a case in (Idczak, 1998, inequality in condition (13) and inequality (25)) the constant π should be replaced by T.*

Remark 11. *It is easy to observe that the results contained in Idczak (1998) remains true with the set $\mathbb{R}^m$ (appearing in the domains of those functions F and F_0) replaced by our set $M \subset \mathbb{R}^m$.*

We say that a pair $(z(\cdot), u(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^\infty$ satisfies (11) if

$$\int_0^T \left\langle H_{\dot{z}}(t, z(t), \dot{z}(t), u(t)), \dot{h}(t) \right\rangle dt = -\int_0^T \left\langle H_z(t, z(t), \dot{z}(t), u(t)), h(t) \right\rangle dt$$

for any $h(\cdot) \in H_0^1((0, T), \mathbb{R}^n)$.

Let us also consider the following two auxiliary problems (below, $b \in \mathbb{R}^n$ is a fixed point):

$$\frac{d}{dt}(F_{\dot{x}}(t, x(t), \dot{x}(t), u(t))) = F_x(t, x(t), \dot{x}(t), u(t)), \ t \in (0, T) \text{ a.e.,} \tag{14}$$

$$x(0) = 0, \ x(T) = b \tag{15}$$

$$J_T(x(\cdot), u(\cdot)) = \int_0^T F_0(t, x(t), \dot{x}(t), u(t))dt \to \min. \tag{16}$$

and problem (11)-(13) with functions H, H_0 given by $H(t, z, \dot{z}, u) = F(t, z + \frac{b}{T}t, \dot{z} + \frac{b}{T}, u)$, $H_0(t, z, \dot{z}, u) = F_0(t, z + \frac{b}{T}t, \dot{z} + \frac{b}{T}, u)$, i.e.

$$\frac{d}{dt}(F_{\dot{z}}(t, z(t) + \frac{b}{T}t, \dot{z}(t) + \frac{b}{T}, u(t))) = F_z(t, z(t) + \frac{b}{T}t, \dot{z}(t) + \frac{b}{T}, u(t)), \ t \in (0, T) \text{ a.e.,} \tag{17}$$

$$z(0) = 0, \ z(T) = 0 \tag{18}$$

$$I_T(z(\cdot), u(\cdot)) = \int_0^T F_0(t, z(t) + \frac{b}{T}t, \dot{z}(t) + \frac{b}{T}, u(t))dt \to \min. \tag{19}$$

We say that a pair $(x_0(\cdot), u_0(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^{\infty}$ $((z_0(\cdot), u_0(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^{\infty})$ is the solution to problem (14)-(16) ((11)-(13)) if it satisfies (14)-(15) ((11)-(12)) and

$$\int_0^T F_0(t, x_0(t), \dot{x}_0(t), u_0(t))dt \leq \int_0^T F_0(t, x(t), \dot{x}(t), u(t))dt$$

$$\left(\int_0^T H_0(t, z_0(t), \dot{z}_0(t), u_0(t))dt \leq \int_0^T H_0(t, z(t), \dot{z}(t), u(t))dt \right)$$

for any pair $(x(\cdot), u(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^{\infty}$ satisfying (14)-(15) $((z(\cdot), u(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^{\infty}$ satisfying (11)-(12)).

In the proof of the maximum principle we shall use the following lemma.

Lemma 12. *If a pair $(x_0(\cdot), u_0(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^{\infty}$ is the solution to problem (14)-(16), then the pair $(z_0(\cdot), u_0(\cdot))$ where $z_0(t) = x_0(t) - \frac{b}{T}t$ for $t \in (0, T)$, is the solution to problem (17)-(19).*

Proof. Let as suppose that the pair $(z_0(\cdot), u_0(\cdot))$ given in the Lemma is not the solution to problem (17)-(19). So, there exists a pair $(z_*(\cdot), u_*(\cdot)) \in H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^{\infty}$ satisfying (17)-(18) and such that

$$\int_0^T F_0(t, z_*(t) + \frac{b}{T}t, \dot{z}_*(t) + \frac{b}{T}, u_*(t))dt < \int_0^T F_0(t, z_0(t) + \frac{b}{T}t, \dot{z}_0(t) + \frac{b}{T}, u_0(t))dt.$$

But then the pair $(x_*(\cdot), u_*(\cdot))$ where $x_*(t) = z_*(t) + \frac{b}{T}t$ for $t \in (0, T)$, belongs to $H^1((0, T), \mathbb{R}^n) \times \mathcal{U}_{(0,T)}^{\infty}$, satisfies (14)-(15) and

$$\int_0^T F_0(t, x_*(t), \dot{x}_*(t), u_*(t))dt = \int_0^T F_0(t, z_*(t) + \frac{b}{T}t, \dot{z}_*(t) + \frac{b}{T}, u_*(t))dt$$

$$< \int_0^T F_0(t, z_0(t) + \frac{b}{T}t, \dot{z}_0(t) + \frac{b}{T}, u_0(t))dt = \int_0^T F_0(t, x_0(t), \dot{x}_0(t), u_0(t))dt.$$

This contradicts the optimality of the pair $(x_0(\cdot), u_0(\cdot))$. ∎

5.1 General case

Now, we shall prove

Theorem 13 (maximum principle I). *Let assumptions A1a, A1b, A2, A3, A4b and A7 be satisfied (without A7c, A7d, A7f - cf. assumption B3 given below). Additionally, assume that G is twicely differentiable in $x \in \mathbb{R}^n$ and*

> *B1. function $G_{xx}(t, \cdot, \cdot) : \mathbb{R}^n \times M \to \mathbb{R}^{n \times n}$ is continuous for $t \in I$ a.e., function $G_{xx}(\cdot, x, u) : I \to \mathbb{R}^{n \times n}$ is measurable in Lebesgue sense for any $(x, u) \in \mathbb{R}^n \times M$*

> *B2. there exist a constant $d_1 > 0$, functions $d_0(\cdot) \in L^2(I, \mathbb{R})$, $e_0(\cdot) \in L^2_{loc}(I, \mathbb{R})$ and a continuous function $a : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that*
>
> $$|G_x(t, x, u)| \leq d_1 |x| + d_0(t),$$
>
> $$|G_{xx}(t, x, u)| \leq a(|x|)e_0(t)$$
>
> *for $t \in I$ a.e., $x \in \mathbb{R}^n$, $u \in M$*

> *B3. functions $f(t, \cdot, \cdot) : \mathbb{R}^n \times M \to \mathbb{R}$, $f_x(t, \cdot, \cdot) : \mathbb{R}^n \times M \to \mathbb{R}^n$ are continuous for $t \in I$ a.e.; functions $f(\cdot, x, u) : I \to \mathbb{R}$, $f_x(\cdot, x, u) : I \to \mathbb{R}^n$ are measurable in Lebesgue sense for any $(x, u) \in \mathbb{R}^n \times M$ and there exist a function $\beta \in L^1(I, \mathbb{R})$, a continuous function $b : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and a function $\gamma(\cdot) \in L^1_{loc}(I, \mathbb{R})$ such that*
>
> $$|f(t, x, u)| \leq \beta(t),$$
>
> $$|f_x(t, x, u)| \leq b(|x|)\gamma(t)$$
>
> *for $t \in I$ a.e., $x \in \mathbb{R}^n$, $u \in M$.*

Let us also assume that, for $t \in I$ a.e., $x \in \mathbb{R}^n$, $u \in M$, the set

$$\{(G_x(t, x, u), f(t, x, u)) \in \mathbb{R}^n \times \mathbb{R}; \ u \in M\}$$

is convex.

If a pair $(x_0(\cdot), u_0(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ is a solution to problem (1)-(6) in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$ according to any definition of optimality given in Section 4, then for any $T > 0$ there exists a function $\lambda_T \in H_0^1((0, T), \mathbb{R}^n)$ such that

$$\frac{d}{dt}(\alpha(t) + \dot{\lambda}_T(t)) = f_x(t, x_0(t), u_0(t)) + G_{xx}(t, x_0(t), u_0(t))\lambda_T(t), \ t \in (0, T) \ a.e. \quad (20)$$

and

$$f(t, x_0(t), u_0(t)) + \langle G_x(t, x_0(t), u_0(t)), \lambda_T(t) \rangle = \min_{u \in M}\{f(t, x_0(t), u) + \langle G_x(t, x_0(t), u), \lambda_T(t) \rangle\}$$
$$(21)$$

for $t \in (0, T)$ a.e.

Proof. Using the optimality principle (if it is needed) we assert that the pair $(x_0(\cdot), u_0(\cdot))$ is finitely optimal solution to problem (1)-(6) in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^\infty$. Let us fix any $T > 0$. So, the pair $(x_0(\cdot)\big|_{(0,T)}, u_0(\cdot)\big|_{(0,T)})$ is the solution to problem (14)-(16) with $b = x_0(T)$ and

$$F(t, x, \dot{x}, u) = \frac{1}{2}\left|\dot{x}\right|^2 + G(t, x, u),$$

$$F_0(t, x, \dot{x}, u) = \left\langle \alpha(t), \dot{x} \right\rangle + f(t, x, u).$$

From Lemma 12 it follows that the pair $(z_0(\cdot), u_0(\cdot))$ where $z_0(t) = x_0(t) - \frac{x_0(T)}{T}t$ for $t \in (0, T)$, is the solution to problem (17)-(19). This means that this pair is the solution to problem (11)-(13) with

$$
\begin{aligned}
H(t, z, \dot{z}, u) &= F(t, z + \frac{x_0(T)}{T}t, \dot{z} + \frac{x_0(T)}{T}, u) = \frac{1}{2}\left|\dot{z} + \frac{x_0(T)}{T}\right|^2 + G(t, z + \frac{x_0(T)}{T}t, u) \\
&= \frac{1}{2}|\dot{z}|^2 + \frac{1}{T}\left\langle \dot{z}, x_0(T)\right\rangle + \frac{1}{T^2}|x_0(T)|^2 + G(t, z + \frac{x_0(T)}{T}t, u),
\end{aligned}
$$

$$
\begin{aligned}
H_0(t, z, \dot{z}, u) &= F_0(t, z + \frac{x_0(T)}{T}t, \dot{z} + \frac{x_0(T)}{T}, u) = \left\langle \alpha(t), \dot{z} + \frac{x_0(T)}{T}\right\rangle + f(t, z + \frac{x_0(T)}{T}t, u) \\
&= \left\langle \alpha(t), \dot{z}\right\rangle + \frac{1}{T}\left\langle \alpha(t), x_0(T)\right\rangle + f(t, z + \frac{x_0(T)}{T}t, u).
\end{aligned}
$$

It is easy to check that the functions H, H_0 satisfies all of the assumptions of the maximum principle proved in Idczak (1998) (cf. Remarks 11, 14). Consequently, there exists a function $\lambda_T \in H_0^1((0, T), \mathbb{R}^n)$ such that

$$
\frac{d}{dt}(\alpha(t) + \dot{\lambda}_T(t)) = f_x(t, z_0(t) + \frac{x_0(T)}{T}t, u_0(t)) + G_{xx}(t, z_0(t) + \frac{x_0(T)}{T}t, u_0(t))\lambda_T(t)
$$

for $t \in (0, T)$ a.e. and

$$
\begin{aligned}
&\left\langle \alpha(t), \dot{z}_0(t)\right\rangle + \frac{1}{T}\left\langle \alpha(t), x_0(T)\right\rangle + f(t, z_0(t) + \frac{x_0(T)}{T}t, u_0(t)) \\
&+ \left\langle G_x(t, z_0(t) + \frac{x_0(T)}{T}t, u_0(t)), \lambda_T(t)\right\rangle + \left\langle \dot{z}_0(t) + \frac{1}{T}x_0(T), \dot{\lambda}_T(t)\right\rangle \\
&= \min_{u \in M}\{\left\langle \alpha(t), \dot{z}_0(t)\right\rangle + \frac{1}{T}\left\langle \alpha(t), x_0(T)\right\rangle + f(t, z_0(t) + \frac{x_0(T)}{T}t, u) \\
&+ \left\langle G_x(t, z_0(t) + \frac{x_0(T)}{T}t, u), \lambda_T(t)\right\rangle + \left\langle \dot{z}_0(t) + \frac{1}{T}x_0(T), \dot{\lambda}_T(t)\right\rangle\}
\end{aligned}
$$

for $t \in (0, T)$ a.e., i.e. (20) and (21) hold true. ∎

Remark 14. *In this remark we use symbols from Idczak (1998). From assumption A4b it follows that, in our case, the matrix $C(x)$, $x \in (0, T)$ a.e., given in (Idczak, 1998, Lemma 4), is nonnegative. In such a case condition (Idczak, 1998, (25)) can be replaced by the following one*

$$
\inf\{B(x)z'z'; |z'| = 1, x \in S\} - 2T(\operatorname*{ess\,sup}_I |A(x)|) > 0.
$$

In fact, in our case, the matrix $A(x)$, $x \in (0, T)$ a.e., appearing above, is the zero matrix.

5.2 Some special case
Now, we shall prove a maximum principle in the case of integrand f not depending on x.

Theorem 15 (maximum principle II)**.** *Let the assumptions of the previous theorem be satisfied. Additionally, assume that*

C1. *function f does not depend on x*

C2. $\alpha(\cdot) \in L^{\infty}(I, \mathbb{R}^n)$

C3. *function $G_{xx} : (0, \infty) \times \mathbb{R}^n \times M \to \mathbb{R}^{n \times n}$ is bounded and for $t \in I$ a.e., $x \in \mathbb{R}^n$, $u \in M$ the matrix $G_{xx}(t, x, u)$ is nonnegative, i.e.*

$$\langle G_{xx}(t, x, u)\lambda, \lambda \rangle \geq 0$$

for $\lambda \in \mathbb{R}^n$.

Then, if a pair $(x_0(\cdot), u_0(\cdot)) \in H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^{\infty}$ is a solution to problem (1)-(6) in the set $H_0^1(I, \mathbb{R}^n) \times \mathcal{U}^{\infty}$ according to any definition of optimality given in Section 4, then there exists a function $\lambda : I \to \mathbb{R}^n$ such that $\lambda \mid_{(0,T)} \in H^1((0, T), \mathbb{R}^n)$, $\lambda(0) = 0$ and

$$\int_0^{\infty} \left\langle \alpha(t) + \dot{\lambda}(t), \dot{\varphi}(t) \right\rangle dt = \int_0^{\infty} \langle G_{xx}(t, x_0(t), u_0(t))\lambda(t), \varphi(t) \rangle dt \tag{22}$$

for any $\varphi(\cdot) \in C_c^1(I, \mathbb{R}^n)$,

$$f(t, u_0(t)) + \langle G_x(t, x_0(t), u_0(t)), \lambda(t) \rangle = \min_{u \in M}\{f(t, u) + \langle G_x(t, x_0(t), u), \lambda(t) \rangle\}, \ t \in I \ a.e. \tag{23}$$

Proof. Let us consider equation (22) on an interval $(0, T)$ (with a fixed $T > 0$), i.e.

$$\frac{d}{dt}(\alpha(t) + \dot{\lambda}(t)) = G_{xx}(t, x_0(t), u_0(t))\lambda(t), \ t \in (0, T) \ a.e., \tag{24}$$

with boundary conditions $\lambda(0) = \lambda(T) = 0$. It is easy to see that it is the Euler-Lagrange equation for the functional

$$\mathcal{F} : H_0^1((0, T), \mathbb{R}^n) \to \mathbb{R},$$

$$\mathcal{F}(\lambda(\cdot)) = \int_0^T (\frac{1}{2}\left|\dot{\lambda}(t)\right|^2 + \left\langle \alpha(t)\dot{\lambda}(t) \right\rangle + \frac{1}{2}\langle \gamma(t)\lambda(t), \lambda(t) \rangle)dt = \int_0^T K(t, \lambda(t), \dot{\lambda}(t))dt$$

where $\gamma(t) = G_{xx}(t, x_0(t), u_0(t))$ and $K(t, \lambda, \dot{\lambda}) = \frac{1}{2}(\dot{\lambda})^2 + \left\langle \alpha(t)\dot{\lambda} \right\rangle + \frac{1}{2}\langle \gamma(t)\lambda, \lambda \rangle$. The function K satisfies assumptions of (Idczak, 1998, Th. 4) and is strictly convex in $(\lambda, \dot{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^n$. Consequently, the function λ_T from Theorem 13 is the unique minimum point of $\mathcal{F}$. So,

$$\mathcal{F}(\lambda_T(\cdot)) < \mathcal{F}(0) = 0$$

In the same way as in Walczak (1995) we check that

$$\mathcal{F}(\lambda(\cdot)) \geq \frac{1}{2}\|\lambda(\cdot)\|^2_{H_0^1((0,T),\mathbb{R}^n)} - \|\alpha(\cdot)\|_{L^2(I,\mathbb{R}^n)}\|\lambda(\cdot)\|_{H_0^1((0,T),\mathbb{R}^n)}.$$

The last two inequalities imply that

$$\left(\int_0^T \left|\dot{\lambda}_T(t)\right|^2 dt\right)^{\frac{1}{2}} = \|\lambda_T(\cdot)\|_{H_0^1((0,T),\mathbb{R}^n)} \leq 2\|\alpha(\cdot)\|_{L^2(I,\mathbb{R}^n)}.$$

for any $T > 0$.

Now, let us put $T_n = n$, $n \in \mathbb{N}$, and consider a sequence $(\lambda_n(\cdot))_{n\in\mathbb{N}}$ of solutions $\lambda_n(\cdot) = \lambda_{T_n}(\cdot)$ to system (24), belonging to $H_0^1((0, T_n), \mathbb{R}^n)$, respectively. Next, let us fix an interval $[0, T_1]$ and consider the sequence of functions $(\lambda_n(\cdot) \mid_{[0,T_1]})_{n\geq 1}$. The last inequality gives

$$\int_0^{T_1} |\lambda_n(t)|^2 \, dt = \int_0^{T_1} \left| \int_0^t \dot{\lambda}_n(s) ds \right|^2 dt \leq T_1 \|\lambda_n(\cdot)\|^2_{H_0^1((0,T_n),\mathbb{R}^n)} \leq T_1 \rho^2, \ n \in \mathbb{N},$$

where $\rho = 2 \|\alpha(\cdot)\|_{L^2(I,\mathbb{R}^n)}$ (the constant not depending on n); of course,

$$\int_0^{T_1} \left| \dot{\lambda}_n(t) \right|^2 dt \leq \rho^2, \ n \in \mathbb{N}.$$

These inequalities mean that the sequence $(\lambda_n(\cdot) \mid_{[0,T_1]})_{n\in\mathbb{N}}$ is bounded in $H^1((0, T_1), \mathbb{R}^n)$. So, from the sequence $(\lambda_n(\cdot))_{n\in\mathbb{N}}$ one can choose a subsequence $(\lambda_n^1(\cdot))_{n\in\mathbb{N}}$ such that the sequence $(\lambda_n^1(\cdot) \mid_{[0,T_1]})_{n\in\mathbb{N}}$ is weakly convergent in $H^1((0, T_1), \mathbb{R}^n)$ to some function $\lambda_{T_1}(\cdot) \in H^1((0, T_1), \mathbb{R}^n)$. From the Arzela-Ascoli theorem and from the uniqueness of the weak limit in the space $C([0, T_1], \mathbb{R}^n)$ of continuous functions on $[0, T_1]$ it follows that we can assume, without loss of the generality, that the sequence $(\lambda_n^1(\cdot) \mid_{[0,T_1]})_{n\in\mathbb{N}}$ converges also uniformly on $[0, T_1]$ to $\lambda_{T_1}(\cdot)$. In particular, $\lambda_{T_1}(0) = 0$.

In the same way (we can assume, without loss of the generality, that the domains of the all functions $\lambda_n^1(\cdot)$, $n \in \mathbb{N}$, contain the interval $[0, T_2]$) from the sequence $(\lambda_n^1(\cdot))_{n\in\mathbb{N}}$ one can choose a subsequence $(\lambda_n^2(\cdot))_{n\in\mathbb{N}}$ such that the sequence $(\lambda_n^2(\cdot) \mid_{[0,T_2]})_{n\in\mathbb{N}}$ is weakly convergent in $H^1((0, T_2), \mathbb{R}^n)$ and uniformly on $[0, T_2]$ to some function $\lambda_{T_2}(\cdot) \in H^1((0, T_2), \mathbb{R}^n)$. Of course, $\lambda_{T_2}(\cdot) \mid_{[0,T_1]} = \lambda_{T_1}(\cdot)$.

Continuing this procedure we obtain a sequence of sequences: $(\lambda_n^1(\cdot))_{n\in\mathbb{N}}$, $(\lambda_n^2(\cdot))_{n\in\mathbb{N}}$, ..., $(\lambda_n^k(\cdot))_{n\in\mathbb{N}}$, ... such that, for any $k \in \mathbb{N}$, the sequence $(\lambda_n^{k+1}(\cdot))_{n\in\mathbb{N}}$ is a subsequence of the sequence $(\lambda_n^k(\cdot))_{n\in\mathbb{N}}$, the sequence $(\lambda_n^k(\cdot) \mid_{[0,T_k]})_{n\in\mathbb{N}}$ is weakly convergent in $H^1((0, T_k), \mathbb{R}^n)$ and uniformly on $[0, T_k]$ to a function $\lambda_{T_k}(\cdot) \in H^1((0, T_k), \mathbb{R}^n)$ and $\lambda_{T_{k+1}}(\cdot) \mid_{[0,T_k]} = \lambda_{T_k}(\cdot)$.

Let $\lambda(\cdot) : I \to \mathbb{R}^n$ be a function such that $\lambda(\cdot) \mid_{[0,T_k]} = \lambda_{T_k}(\cdot)$ for $k \in \mathbb{N}$. It is easy to see that the function $\lambda(\cdot)$ has on I the weak derivative $\dot{\lambda}(\cdot)$ and $\dot{\lambda}(\cdot) \mid_{[0,T_k]} = \left(\lambda(\cdot) \mid_{[0,T_k]} \right)^{\cdot}$. Let us consider the sequence $(\lambda_n^n(\cdot))_{n\in\mathbb{N}}$ and the sequence $(\lambda_n^n(\cdot) \mid_{[0,T_n]})_{n\in\mathbb{N}}$. Let us denote the second sequence by $(\mu_n(\cdot))_{n\in\mathbb{N}}$. On each interval $[0, T_k]$, this sequence (with a precision to the first $(k-1)$ elements) is weakly convergent in $H^1((0, T_k), \mathbb{R}^n)$ and uniformly on $[0, T_k]$ to the function $\lambda(\cdot) \mid_{[0,T_k]} \in H^1((0, T_k), \mathbb{R}^n)$

From the maximum principle I it follows that each function $\mu_n(\cdot)$ satisfies the conditions (domain of $\mu_n(\cdot)$ contains the interval $[0, T_n]$)

$$\frac{d}{dt}(\alpha(t) + \dot{\mu}_n(t)) = G_{xx}(t, x_0(t), u_0(t))\mu_n(t), \ t \in (0, T_n) \text{ a.e.}, \tag{25}$$

$$f(t, u_0(t)) + \langle G_x(t, x_0(t), u_0(t)), \mu_n(t) \rangle = \min_{u \in M}\{f(t, u) + \langle G_x(t, x_0(t), u), \mu_n(t) \rangle\}, \ t \in (0, T_n) \text{ a.e.}, \tag{26}$$

The first condition is equivalent to the following one

$$\int_0^{T_n} \left\langle \alpha(t) + \dot{\mu}_n(t), \dot{\varphi}(t) \right\rangle dt = \int_0^{T_n} \langle G_{xx}(t, x_0(t), u_0(t))\mu_n(t), \varphi(t) \rangle dt$$

for any $\varphi(\cdot) \in C_c^1((0, T_n), \mathbb{R}^n)$ (the space of continuously differentiable functions $\varphi : (0, T_n) \to \mathbb{R}^n$ with compact support $\mathrm{supp}\,\varphi \subset (0, T_n)$).

Now, we shall show that (22) holds true. Indeed, let $\varphi(\cdot) \in C_c^1(I, \mathbb{R}^n)$ and $n_0 \in \mathbb{N}$ be such that $\mathrm{supp}\,\varphi \subset (0, T_{n_0})$. Then

$$\int_0^\infty \left\langle \alpha(t) + \dot{\lambda}(t), \dot{\varphi}(t) \right\rangle dt = \int_0^{T_{n_0}} \left\langle \alpha(t) + \dot{\lambda}(t), \dot{\varphi}(t) \right\rangle dt = \lim_{\substack{n \to \infty \\ n \geq n_0}} \int_0^{T_{n_0}} \left\langle \alpha(t) + \dot{\mu}_n(t), \dot{\varphi}(t) \right\rangle dt$$

$$= \lim_{\substack{n \to \infty \\ n \geq n_0}} \int_0^{T_{n_0}} \langle G_{xx}(t, x_0(t), u_0(t))\mu_n(t), \varphi(t) \rangle dt = \int_0^{T_{n_0}} \langle G_{xx}(t, x_0(t), u_0(t))\lambda(t), \varphi(t) \rangle dt$$

$$= \int_0^\infty \langle G_{xx}(t, x_0(t), u_0(t))\lambda(t), \varphi(t) \rangle dt$$

(the second equality follows from the weak convergence in $H^1((0, T_{n_0}), \mathbb{R}^n)$ of the sequence $(\mu_n(\cdot) \mid_{[0, T_{n_0}]})_{n \in \mathbb{N}}$ to the function $\lambda(\cdot) \mid_{[0, T_{n_0}]}$ and the fourth equality follows from the uniform convergence on $[0, T_{n_0}]$ of the sequence $(\mu_n(\cdot) \mid_{[0, T_{n_0}]})_{n \in \mathbb{N}}$ to the function $\lambda(\cdot) \mid_{[0, T_{n_0}]}$ and Lebesgue bounded convergence theorem). So, (22) holds true.

Now, we shall show that (23) holds true. Indeed, let $Z_n \subset (0, T_n)$, $n \in \mathbb{N}$, be a set of zero measure such that (26) does not hold on it. Let us fix a point $t \in I \setminus \bigcup_{n=1}^\infty Z_n$ and let $n_0 \in \mathbb{N}$ be the smallest positive integer such that $t \in (0, T_{n_0})$. We have

$$f(t, u_0(t)) + \langle G_x(t, x_0(t), u_0(t)), \mu_n(t) \rangle = \min_{u \in M} \{ f(t, u) + \langle G_x(t, x_0(t), u), \mu_n(t) \rangle \}$$

for $n \geq n_0$. Since the functions

$$\varkappa_u : \mathbb{R}^n \ni \mu \longmapsto f(t, u) + \langle G_x(t, x_0(t), u), \mu \rangle \in \mathbb{R}$$

with $u \in M$ are lipschitzian with the same constant (compactness of M and continuity of G_x in $u \in M$ are important here), therefore (cf. (Łojasiewicz, 1988, Part III.2, Th. 1)) the function

$$\mathbb{R}^n \ni \mu \longmapsto \min_{u \in M} \varkappa_u(\mu) = \min_{u \in M} \{ f(t, u) + \langle G_x(t, x_0(t), u), \mu \rangle \}$$

is continuous. Consequently, the fact that $\lim_{\substack{n \to \infty \\ n \geq n_0}} \mu_n(t) = \lambda(t)$ implies (23). ∎

6. Concluding remarks

Main results of the paper are contained in Theorems 9, 13 and 15. In Theorem 9 a connection between the notions of optimality in infinite and finite horizon cases is established. Theorem 13 contains necessary conditions for each of the introduced kinds of optimality in general case and Theorem 15 contains such conditions in some special case.

Open problems are maximum principles (in both, special and general case) stating the existence of a Lagrange multiplier $\lambda : I \to \mathbb{R}^n$ satisfying a conjugate system with the space $C_c^1(I, \mathbb{R}^n)$ replaced by $H_0^1(I, \mathbb{R}^n)$ and the minimum condition a.e. on I.

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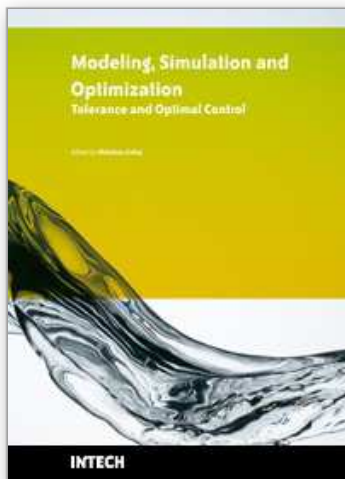
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